

Warped $AdS_6 \times S^2$ in Type IIB supergravity I: local solutions

Eric D'Hoker,^a Michael Gutperle,^a Andreas Karch^b and Christoph F. Uhlemann^b

^a*Department of Physics and Astronomy, University of California Los Angeles,
405 Hilgard Avenue, Los Angeles, CA 90095-1547, U.S.A.*

^b*Department of Physics, University of Washington,
P.O. Box 351560, Seattle, WA 98195-1560, U.S.A.*

E-mail: dhoker@physics.ucla.edu, gutperle@physics.ucla.edu,
akarch@uw.edu, uhlemann@uw.edu

ABSTRACT: We investigate the existence of solutions with 16 residual supersymmetries to Type IIB supergravity on a space-time of the form $AdS_6 \times S^2$ warped over a two-dimensional Riemann surface Σ . The $SO(2, 5) \times SO(3)$ isometry extends to invariance under the exceptional Lie superalgebra $F(4)$. In the present paper, we construct the general Ansatz compatible with these symmetries, derive the corresponding reduced BPS equations, and obtain their complete local solution in terms of two locally holomorphic functions \mathcal{A}_\pm on Σ , subject to certain positivity and regularity conditions. Globally, $(\mathcal{A}_+, \mathcal{A}_-)$ are allowed to be multiple-valued on Σ and be holomorphic sections of a holomorphic bundle over Σ with structure group contained in $SU(1, 1) \times \mathbb{C}$. Globally regular solutions are expected to provide the near-horizon geometry of (p, q) 5-brane and 7-brane webs which are holographic duals to five-dimensional conformal field theories. A preliminary analysis of the positivity and regularity conditions will be presented here, leaving the construction of globally regular solutions to a subsequent paper.

KEYWORDS: D-branes, Gauge-gravity correspondence

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1 Introduction

Gauge-gravity duality, namely the equivalence between a quantum field theory in d dimensions and a gravitational theory in $d + 1$ dimensions via holography, has become one of the cornerstones of modern theoretical physics. The best understood examples are provided by the AdS/CFT correspondence and involve conformal field theories (CFTs) with a large number of fields and their anti-de Sitter (AdS) dual space-times with large degrees of supersymmetry. Maldacena's original paper [1] contained examples of such equivalences in $d = 2, 3, 4$ and 6 and many more dual pairs have been uncovered in these dimensions since then. However, what is almost entirely missing to date is any well understood example of the AdS/CFT correspondence in dimension $d = 5$, namely between a 4+1 dimensional CFT and its 5+1 dimensional dual AdS space-time.

One reason that $d = 5$ is special is the absence of maximally supersymmetric theories, in which the 16 Poincaré supersymmetries of a 4+1 dimensional CFT would be enhanced by 16 conformal supersymmetries to the maximum allowed number of 32 supersymmetries. The complete classification of superconformal algebras [2] indeed shows that $d = 5$ is singled out and does not support a maximally supersymmetric CFT. Unlike its lower dimensional cousins, supersymmetric Yang-Mills theory with maximal Poincaré supersymmetry in 4+1 dimensions is not a conformal field theory at the origin of its moduli space. Instead, the theory is believed to flow in the ultraviolet (UV) to the 5+1 dimensional $(2, 0)$ theory, a theory which itself features prominently in AdS/CFT (see for example [3]).

What does exist, however, are supersymmetric CFTs (SCFTs) in 4+1 dimensions with 8 Poincaré supersymmetries which are enhanced by 8 conformal supersymmetries to a total of 16 supersymmetries. The superconformal algebra in this case is based on the exceptional Lie superalgebra $F(4)$. This Lie superalgebra has a real form whose maximal bosonic subalgebra is $SO(2,5) \oplus SO(3)$ [4], which may be viewed as the direct sum of the conformal algebra in 4+1 dimensions [4] and an R-symmetry. Explicit field theory examples realizing this algebra have been uncovered in [5, 6]. They are based on supersymmetric gauge theories which flow to a strongly coupled CFT in the UV, as long as the number of matter fields coupled to the gauge field remains sufficiently small. The upper bound on the number of matter fields depends on the gauge group and on the representations of the matter content and can generically be determined from the 1-loop running of the gauge coupling [6]. Since these supersymmetric theories allow for a large N limit, one may expect that they will possess a holographic AdS_6 dual.

String theory realizations of these 4+1 dimensional CFTs via brane embeddings are known and can serve as a natural starting point to construct their AdS_6 duals. For example, 4+1 dimensional SCFTs based on the gauge group $Sp(N)$, with hypermultiplets in the fundamental and anti-symmetric tensor representations of $Sp(N)$, were realized in [5] via a Type IIA string theory construction involving a stack of N D4 branes near a collection of O8 planes and D8 branes. A Type IIA supergravity solution based on this construction was given in [7] and generalized to quiver gauge groups in [8]. While some interesting questions can be addressed in this geometry, its construction leads to singularities, because the presence of the O8 planes forces the dilaton to blow up.

A much more general brane realization of these 4+1 dimensional CFTs can be given via (p, q) brane webs [9, 10] in Type IIB string theory. In these brane webs, D5 branes are suspended between NS5 branes, giving a 4+1 dimensional version of the construction pioneered in [11] for 2+1 dimensional gauge theories. The brane webs can realize the gauge theory at any point on its moduli space, as well as in the presence of relevant deformations such as mass terms or a finite bare coupling. In the limit when all the branes essentially lie on top of one another the webs do realize large classes of the 4+1 dimensional SCFTs of [6], most naturally those based on $SU(N)$ gauge theories, possibly with couplings to matter in the fundamental representation.¹ D7-branes can be added to the (p, q) webs [13], thereby slightly expanding the set of SCFTs that can be realized. As we will see, the most general supergravity Ansatz respecting the symmetries of the (p, q) web does allow for a non-trivial axion winding which is required to accommodate D7-branes. When no D7-branes are present, the (p, q) webs form a special sub-class for which the axion winding vanishes.

It is our goal to construct Type IIB supergravity solutions which are holographically dual to the 4+1-dimensional CFTs realized via (p, q) webs in the large N limit. This will require solving the difficult problem of obtaining fully localized solutions for the corresponding intersecting branes. Examples where such solutions can give rise to warped AdS

¹Alternatively these SCFTs can be realized in terms of M-theory compactifications on singular Calabi-Yau (CY) 3-folds [6]. For the special case of toric CYs, this construction is T-dual to the (p, q) web [12]. In any case, the realization in terms of branes serves as a more natural starting point for holographic considerations, where field theory and AdS gravity are seen as two equivalent descriptions of a given brane setup.

spaces were given in [14, 15]. Our approach will be to use the general Ansatz for the Type IIB fields consistent with the symmetries, reduce the BPS equations to this Ansatz, and then solve the BPS equations explicitly.

Earlier attempts at using the BPS equations appeared in [16–18], where reduced BPS equations were obtained, but not generally solved. A Type IIB T-dual configuration of the D4/D8 solution in Type IIA [7] was used to test these equations. This T-dual solution is even more troublesome than the original Type IIA solution. In addition to the singularity caused by the presence of the O8 plane on the IIA side, one now has an additional singularity due to T-dualizing a U(1) subgroup of the SO(4) isometry associated with an internal sphere. The circle that is being T-dualized shrinks to zero size at the poles of the sphere, giving rise to a new singularity in the Type IIB solution. Nevertheless, this T-dual to the D4/D8 system provides a useful check on the BPS equations obtained in [16–18] and we will use it for a similar purpose.

In the present paper, we will construct the general Ansatz in Type IIB consistent with the $SO(2, 5) \times SO(3)$ symmetries, derive the reduced BPS equations, and construct their general *local solutions* in terms of two locally holomorphic² functions \mathcal{A}_\pm on a Riemann surface Σ . To connect these local supergravity solutions to the CFTs originating from the (p, q) brane system requires that we impose the necessary physical regularity conditions on the supergravity fields of the solutions. For this purpose, Σ must be compact, with or without boundary, and geodesically complete. The conditions on \mathcal{A}_\pm needed to guarantee the proper Minkowskian signature of the metric are local on Σ and given by the following inequalities,

$$\begin{aligned} 0 &< -|\partial_w \mathcal{A}_+|^2 + |\partial_w \mathcal{A}_-|^2 \\ 0 &< |\mathcal{A}_+|^2 - |\mathcal{A}_-|^2 + \mathcal{B} + \bar{\mathcal{B}} \end{aligned} \quad (1.1)$$

where ∂_w is the derivative on Σ with respect to a local holomorphic coordinate w , and \mathcal{B} is defined, up to an additive constant, by the relation $\partial_w \mathcal{B} = \mathcal{A}_+ \partial_w \mathcal{A}_- - \mathcal{A}_- \partial_w \mathcal{A}_+$. As a result, \mathcal{B} is also locally holomorphic. If Σ has a boundary then the two inequalities of (1.1) must hold strictly in the interior of Σ , and become equalities on the boundary of Σ .

The group $SU(1, 1)$ acting linearly on the doublet $(\mathcal{A}_+, \mathcal{A}_-)$ leaves the conditions (1.1) invariant and induces the standard $SU(1, 1) = SL(2, \mathbb{R})$ duality transformations on the supergravity fields. Additionally, the supergravity solutions are invariant under constant shifts $\mathcal{A}_\pm \rightarrow \mathcal{A}_\pm + a_\pm$ for constants satisfying $a_- = \bar{a}_+$, which form the additive group isomorphic to \mathbb{C} . The $SL(2, \mathbb{Z})$ duality symmetry of Type IIB string theory allows us to consider supergravity solutions with identifications under $SL(2, \mathbb{Z})$, namely with non-trivial axion winding number. Mathematically, the problem then becomes to obtain holomorphic sections $(\mathcal{A}_+, \mathcal{A}_-)$ of a holomorphic bundle over Σ with structure group contained in $SL(2, \mathbb{Z}) \times \mathbb{C}$, subject to the positivity conditions of (1.1).

In the simplest case where the Riemann surface Σ is compact and has no boundary, the second inequality in (1.1) becomes trivial since the arbitrary constant in \mathcal{B} can always be chosen to satisfy the inequality. The associated mathematical problem then also simplifies,

²Throughout, we will use the terminology *locally holomorphic* to include the possibility that \mathcal{A}_\pm may be multiple-valued on Σ , and/or have poles or other singularities on the boundary of Σ .

and may be formulated directly in terms of a holomorphic bundle of one-forms $\partial_w \mathcal{A}_\pm$ satisfying the first relation in (1.1). In this paper, we will provide a preliminary analysis into the existence of such global solutions but leave a detailed investigation for future work.

We close the introduction with some remarks on the relation of this work to other investigations into half-BPS solutions to Type IIB, M-theory, and six-dimensional supergravities on space-times built as products of an AdS space and one or several spheres warped over a Riemann surface Σ . In each case, the isometries of the AdS space and the sphere factors are used to reduce the BPS equations to a complicated set of non-linear partial differential equations on Σ , which can be solved exactly in terms of harmonic or holomorphic data on Σ . This strategy was employed successfully to the construction of a large variety of novel supergravity solutions in different contexts. Type IIB supergravity duals to four-dimensional $\mathcal{N} = 4$ super-Yang Mills theory were found in the presence of a planar interface [19] giving supersymmetric Janus field theories [20, 21] and in the presence of Wilson loops [22]. M-theory duals to field theories in three and six dimensions were found in the presence of various defect operators in [23–25]. Finally, six-dimensional supergravity duals were found to two-dimensional conformal field theories with string junctions in [26].³

The unifying principle of this strategy was explained in [32] as follows. The integrability conditions on the BPS equations produce Bianchi and field equations. With enough supersymmetry, all the Bianchi and field equations may be obtained as integrability conditions on the system of BPS equations, which therefore play a role somewhat analogous to that of a Lax pair for integrable systems. Upon reduction to $AdS_6 \times S^2$ warped over Σ , the BPS equations reduce to equations for functions on Σ and genuinely become a set of Lax equations for a system that must therefore be integrable in the classic sense. The conformal invariance of these systems lies at the origin of their solvability in terms of harmonic and holomorphic data on a Riemann surface Σ .

We note that close cousins to the half-BPS solutions to Type IIB supergravity obtained here are the half-BPS solutions on a space-time of the form $AdS_2 \times S^6$ warped over a Riemann surface Σ . The superconformal algebra is now a different real form of the exceptional Lie superalgebra $F(4)$, this time with maximal bosonic subalgebra $SO(2, 1) \times SO(7)$, and 16 supersymmetries as well. The two problems are related just as duals to Wilson loops in [22] are related to the planar interface solutions in [19].

1.1 Organization

The outline of this paper is as follows. In section 2 we will review the basics of Type IIB supergravity and introduce a suitable Ansatz. In section 3 we use the decomposition of the supersymmetry generators onto Killing spinors of AdS_6 and S^2 to reduce the BPS equations for the reduced supergravity fields to a system of algebraic and partial differential equations on the surface Σ , which we partially solve. In section 4 we solve the reduced BPS equations completely and obtain the most general local solutions to the BPS equation in terms of holomorphic data. In section 5 we summarize the expressions for all the supergravity fields in terms of the holomorphic data, analyze their behavior under $SU(1, 1)$ symmetry

³See also [27–31] for related work on these systems using a variety of different strategies.

of Type IIB supergravity, obtain the regularity conditions, recover the singular T-dual of the D4/D8 system, and conclude the section with arguments in favor of monodromy of the holomorphic data. We conclude with a discussion in section 6. In appendix A, a basis for the Dirac-Clifford algebra adapted to our Ansatz is presented, while the geometry of Killing spinors is reviewed in appendix B. Details of the derivation of the BPS equations are in appendix C, of the Bianchi identities in appendix D, and of the expressions for the supergravity fields in terms of holomorphic data are in appendix E.

2 Type IIB supergravity and $AdS_6 \times S^2 \times \Sigma$ Ansatz

In this section, we provide a brief review of the Type IIB supergravity fields, Bianchi identities, field equations and BPS equations, and their $SU(1,1)$ duality symmetry, and go on to construct the general Ansatz for the bosonic fields of the solutions we seek to construct. As laid out in the introduction, the bosonic symmetries of the Ansatz are completely determined by the superconformal algebra.

2.1 Type IIB supergravity review

The bosonic fields of Type IIB supergravity [33, 34] consist of the metric g_{MN} , a one-form P and gauge connection Q representing the axion-dilaton field strengths, a complex three-form field strength G , and a self-dual five-form field strength F_5 . The fields satisfy the following Bianchi identities,

$$\begin{aligned} 0 &= dP - 2iQ \wedge P \\ 0 &= dQ + iP \wedge \bar{P} \\ 0 &= dG - iQ \wedge G + P \wedge \bar{G} \\ 0 &= 8dF_{(5)} - iG \wedge \bar{G} \end{aligned} \tag{2.1}$$

The field strength $F_{(5)}$ is required to be self-dual,

$$F_{(5)} = *F_{(5)} \tag{2.2}$$

The field equations are given by,

$$\begin{aligned} 0 &= \nabla^M P_M - 2iQ^M P_M + \frac{1}{24}G_{MNP}G^{MNP} \\ 0 &= \nabla^P G_{MNP} - iQ^P G_{MNP} - P^P \bar{G}_{MNP} + \frac{2}{3}iF_{(5)MNPQR}G^{PQR} \\ 0 &= R_{MN} - P_M \bar{P}_N - \bar{P}_M P_N - \frac{1}{6}(F_{(5)}^2)_{MN} \\ &\quad - \frac{1}{8}(G_M{}^{PQ}\bar{G}_{NPQ} + \bar{G}_M{}^{PQ}G_{NPQ}) + \frac{1}{48}g_{MN}G^{PQR}\bar{G}_{PQR} \end{aligned} \tag{2.3}$$

The fermionic fields are the dilatino λ and the gravitino ψ_M , both of which are complex Weyl spinors with opposite 10-dimensional chiralities, given by $\Gamma_{11}\lambda = \lambda$, and $\Gamma_{11}\psi_M =$

$-\psi_M$. The supersymmetry variations of the fermions are,

$$\begin{aligned}\delta\lambda &= i(\Gamma \cdot P)\mathcal{B}^{-1}\varepsilon^* - \frac{i}{24}(\Gamma \cdot G)\varepsilon \\ \delta\psi_M &= D_M\varepsilon + \frac{i}{480}(\Gamma \cdot F_{(5)})\Gamma_M\varepsilon - \frac{1}{96}(\Gamma_M(\Gamma \cdot G) + 2(\Gamma \cdot G)\Gamma_M)\mathcal{B}^{-1}\varepsilon^*\end{aligned}\quad (2.4)$$

where \mathcal{B} is the charge conjugation matrix of the Dirac-Clifford algebra.⁴ The BPS equations are obtained by setting $\delta\lambda = \delta\psi_M = 0$.

The Bianchi identities (2.1) for the field strengths $P, Q, G, F_{(5)}$ can be solved in terms of a complex scalar B ; a complex 2-form potential $C_{(2)}$, and a real 4-form potential $C_{(4)}$. The fields P and Q are expressed as follows,

$$\begin{aligned}P &= f^2 dB & f^2 &= (1 - |B|^2)^{-1} \\ Q &= f^2 \text{Im}(B d\bar{B})\end{aligned}\quad (2.5)$$

while the fields G and $F_{(5)}$ are conveniently expressed in terms of $C_{(2)}$ and $C_{(4)}$ with the help of the complex field strength $F_{(3)} = dC_{(2)}$,

$$\begin{aligned}G &= f(F_{(3)} - B\bar{F}_{(3)}) \\ F_{(5)} &= dC_{(4)} + \frac{i}{16}(C_{(2)} \wedge \bar{F}_{(3)} - \bar{C}_{(2)} \wedge F_{(3)})\end{aligned}\quad (2.6)$$

The scalar field B is related to the complex scalar τ and the axion χ , and dilaton ϕ by,

$$B = \frac{1 + i\tau}{1 - i\tau} \quad \tau = \chi + ie^{-2\phi} \quad (2.7)$$

The expectation value of $e^{2\phi}$ is related to the string coupling constant.

2.2 SU(1, 1) duality symmetry

Type IIB supergravity is invariant under $\text{SU}(1, 1) = \text{SL}(2, \mathbb{R})$ symmetry. This symmetry leaves the Einstein frame metric g_{MN} as well as the 4-form $C_{(4)}$ invariant, acts on the field B by Möbius transformations, and acts on the 2-form $C_{(2)}$ and its complex conjugate $\bar{C}_{(2)}$ by a linear transformation,

$$\begin{aligned}B &\rightarrow B^s = \frac{uB + v}{\bar{v}B + \bar{u}} \\ C_{(2)} &\rightarrow C_{(2)}^s = uC_{(2)} + v\bar{C}_{(2)}\end{aligned}\quad (2.8)$$

with $u, v \in \mathbb{C}$ and $|u|^2 - |v|^2 = 1$. In this non-linear realization of $\text{SU}(1, 1)$ on B , the field B takes values in the coset $\text{SU}(1, 1)/\text{U}(1)_q$, and the fermions λ and ψ_μ transform linearly

⁴Our convention for the signature of the 10-dimensional space-time metric is $\eta = \text{diag}(- + \cdots +)$; the Dirac-Clifford algebra is defined by the relations $\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}I_{32}$; and the charge conjugation matrix \mathcal{B} is defined by the relations $\mathcal{B}\mathcal{B}^* = I$ and $\mathcal{B}\Gamma^M\mathcal{B}^{-1} = (\Gamma^M)^*$. We will use the convention that repeated indices are to be summed; complex conjugation will be denoted by *bar* for functions and by *star* for spinors; and we will use the notation $\Gamma \cdot T \equiv \Gamma^{M_1 \cdots M_p} T_{M_1 \cdots M_p}$ for the contraction of any antisymmetric tensor field T of rank p and the Γ -matrix of the same rank.

under the isotropy gauge group $U(1)_q$ with composite gauge field Q . The transformation rules for the field strengths are [33],

$$\begin{aligned} P &\rightarrow P^s = e^{2i\theta} P \\ Q &\rightarrow Q^s = Q + d\theta \\ G &\rightarrow G^s = e^{i\theta} G \end{aligned} \quad (2.9)$$

where the phase θ is defined by,

$$e^{i\theta} = \left(\frac{v\bar{B} + u}{\bar{v}B + \bar{u}} \right)^{\frac{1}{2}} \quad (2.10)$$

The $SU(1, 1) = SL(2, \mathbb{R})$ symmetry will serve as a useful guide to organize the holomorphic data in our local solution. As is well-known, the invariance of Type IIB supergravity under the continuous group $SL(2, \mathbb{R})$ is reduced in Type IIB string theory to invariance under the discrete $SL(2, \mathbb{Z})$ S-duality symmetry, due to the charge quantization of non-perturbative one-branes, five-branes and D-instantons. In the construction of supergravity solutions, we will always allow for the continuous symmetry.

2.3 The $AdS_6 \times S^2 \times \Sigma$ Ansatz

We seek a general Ansatz in Type IIB supergravity with the following symmetry group,

$$SO(2, 5) \times SO(3) \quad (2.11)$$

The factor $SO(2, 5)$ requires the geometry to contain AdS_6 , while the factor $SO(3)$ requires S^2 , so that our space-time is given by,

$$AdS_6 \times S^2 \times \Sigma \quad (2.12)$$

Here Σ stands for the remaining two-dimensional space over which the product $AdS_6 \times S^2$ is warped. In order for the above space to be a Type IIB supergravity geometry, Σ must carry an orientation as well as a Riemannian metric, and is therefore a Riemann surface, possibly with boundary. This $SO(2, 5) \times SO(3)$ -invariant Ansatz for the metric can be written as,

$$ds^2 = f_6^2 ds_{AdS_6}^2 + f_2^2 ds_{S^2}^2 + ds_\Sigma^2 \quad (2.13)$$

where f_6, f_2 and ds_Σ^2 are functions of Σ only. We introduce an orthonormal frame,

$$\begin{aligned} e^m &= f_6 \hat{e}^m & m &= 0, 1, 2, 3, 4, 5 \\ e^i &= f_2 \hat{e}^i & i &= 6, 7 \\ e^a & & a &= 8, 9 \end{aligned} \quad (2.14)$$

where \hat{e}^m and \hat{e}^i respectively refer to the orthonormal frames for the spaces AdS_6 and S^2 with unit radius, and e^a is an orthonormal frame for the metric on Σ . In particular, we have,

$$\begin{aligned} ds_{AdS_6}^2 &= \eta_{mn}^{(6)} \hat{e}^m \otimes \hat{e}^n & \eta^{(6)} &= \text{diag}(- + + + + +) \\ ds_{S^2}^2 &= \delta_{ij} \hat{e}^i \otimes \hat{e}^j \\ ds_\Sigma^2 &= \delta_{ab} e^a \otimes e^b \end{aligned} \quad (2.15)$$

By $SO(2, 5) \times SO(3)$ -invariance, the fields P, Q, G and $F_{(5)}$ are given as follows,

$$\begin{aligned} P &= p_a e^a & G &= g_a e^a \wedge e^{67} \\ Q &= q_a e^a & F_{(5)} &= 0 \end{aligned} \quad (2.16)$$

where $e^{67} = e^6 \wedge e^7$. The components p_a, q_a , and g_a are complex. Note that the Bianchi identity for the five-form field (2.1) is automatically satisfied with this Ansatz.

3 Reducing the BPS equations

The residual supersymmetries, if any, of a configuration of purely bosonic Type IIB supergravity fields are governed by the BPS equations of (2.4). Our interest is in purely bosonic field configurations which preserve 16 independent supersymmetries given by the $AdS_6 \times S^2 \times \Sigma$ Ansatz of the preceding section. It will turn out that any such configuration automatically solves the Bianchi and field equations, and thus automatically provides a half-BPS solution to Type IIB supergravity.

In this section, we will reduce the BPS equations to the $AdS_6 \times S^2 \times \Sigma$ Ansatz by decomposing the supersymmetry parameter ε of (2.4) onto the Killing spinors of $AdS_6 \times S^2$. We will expose the residual symmetries of the reduced BPS equations, and solve those reduced equations which are purely algebraic in the supersymmetry spinor components. This will produce simple algebraic expressions for the metric factors f_2, f_6 in terms of the spinors. The remaining reduced BPS equations will then gradually be solved for the remaining bosonic fields as well as for the residual supersymmetries in subsequent sections. The strategy employed here is very similar to the one used in [19] and so our discussion will closely follow that work.

3.1 Killing spinors

The Killing spinor equations on $AdS_6 \times S^2$ are,⁵

$$\begin{aligned} \left(\hat{\nabla}_m - \frac{1}{2} \eta_1 \gamma_m \otimes I_2 \right) \chi_\alpha^{\eta_1, \eta_2} &= 0 \\ \left(\hat{\nabla}_i - \frac{i}{2} \eta_2 I_8 \otimes \gamma_i \right) \chi_\alpha^{\eta_1, \eta_2} &= 0 \end{aligned} \quad (3.1)$$

where $\hat{\nabla}_m$ and $\hat{\nabla}_i$ stand for the covariant spinor derivatives respectively on the spaces AdS_6 and S^2 with unit radius. Recall that m, i , and a are all *frame indices*. The spinors $\chi_\alpha^{\eta_1, \eta_2}$ are 16-dimensional, and the parameters η_1 and η_2 can take the values ± 1 . The solutions to these equation are 4-fold degenerate for each value of η_1, η_2 , and this degeneracy will be labeled by the index $\alpha = 1, 2, 3, 4$. The chirality matrices act as follows,

$$\begin{aligned} (\gamma_{(1)} \otimes I_2) \chi_\alpha^{\eta_1, \eta_2} &= \chi_\alpha^{-\eta_1, \eta_2} \\ (I_6 \otimes \gamma_{(2)}) \chi_\alpha^{\eta_1, \eta_2} &= \chi_\alpha^{\eta_1, -\eta_2} \end{aligned} \quad (3.2)$$

⁵The decomposition of the 10-dimensional Dirac-Clifford matrices under the reduction to the $AdS_6 \times S^2 \times \Sigma$ Ansatz, and the details of the Killing spinor equations, are relegated to appendices A and B respectively.

The way these equations should be understood is as follows. We begin with $\eta_1 = \eta_2 = +$, and pick a basis χ_α^{++} for the four-dimensional vector space of spinors for fixed η_1, η_2 such that the action $\gamma_{(1)}$ and $\gamma_{(2)}$ is diagonal. Then, we can simply define the basis for $\chi_\alpha^{\eta_1, \eta_2}$ for the remaining three values of η_1, η_2 by the action of the chirality matrices above.

Since ε^* appears in the fermion variations of (2.4), we also need to understand how to express the complex conjugate spinor in this basis. If $\chi_\alpha^{\eta_1, \eta_2}$ satisfies (3.1), then by complex conjugating the entire first equation and using $(\gamma_m)^* = B_{(1)} \gamma_m B_{(1)}^{-1}$, we conclude that $B_{(1)}^{-1} \otimes I_2 (\chi_\alpha^{\eta_1, \eta_2})^*$ satisfies the same equation, with the same values of η_1, η_2 . Proceeding analogously for the second equation, we conclude that $I_6 \otimes B_{(2)}^{-1} (\chi_\alpha^{\eta_1, \eta_2})^*$ also satisfies the same equation, with the same values of η_1, η_2 . As a result, we must have the following linear relation,

$$(B_{(1)}^{-1} \otimes B_{(2)}^{-1}) (\chi_\alpha^{\eta_1, \eta_2})^* = \sum_{\beta=1}^4 M_{\alpha\beta}^{\eta_1, \eta_2} \chi_\beta^{\eta_1, \eta_2} \quad (3.3)$$

for some matrix $M_{\alpha\beta}$ for each pair η_1, η_2 .

We will now show that one may choose a basis for the Killing spinors $\chi_\alpha^{\eta_1, \eta_2}$ in which $M = I$. Iterating the complex conjugation, we conclude that $(M^{\eta_1, \eta_2})^* (M^{\eta_1, \eta_2}) = I$, for all values of η_1, η_2 . Specializing first to $\eta_1 = \eta_2 = +$, we have a single matrix $M^{+,+}$ satisfying $(M^{+,+})^* (M^{+,+}) = I$. Now every such matrix may be rotated to the identity by a general linear complex-valued 4×4 matrix U , using the relation $M^{+,+} = (U^*)^{-1} U$. An easy way to construct U is as follows. An arbitrary invertible complex matrix M in $\text{GL}(4, \mathbb{C})$ may be written as an exponential, $M = \exp(iH)$ of a complex matrix H . Given M , the matrix H is not unique. The condition $M^* M = I$ requires H to be real-valued. Thus, we choose the solution $U = \exp(\frac{i}{2} H)$ to the equation $M^{+,+} = (U^*)^{-1} U$, and the relation for $\eta_1 = \eta_2 = +$ may be diagonalized as follows,

$$(B_{(1)}^{-1} \otimes B_{(2)}^{-1}) (\chi_\alpha^{+,+})^* = \chi_\alpha^{+,+} \quad (3.4)$$

for $\alpha = 1, 2, 3, 4$. For the other values of η_1, η_2 we use (3.2) to express $\chi_\alpha^{\eta_1, \eta_2}$ in terms of $\chi_\alpha^{+,+}$,

$$\begin{aligned} \chi_\alpha^{+,+} &= (\gamma_{(1)} \otimes I_2) \chi_\alpha^{-,+} \\ &= (I_6 \otimes \gamma_{(2)}) \chi_\alpha^{+,-} \\ &= (\gamma_{(1)} \otimes \gamma_{(2)}) \chi_\alpha^{-,-} \end{aligned} \quad (3.5)$$

Using the fact that $\gamma_{(1)}$ commutes with $B_{(1)}$, while $\gamma_{(2)}$ anti-commutes with $B_{(2)}$, we find,

$$(B_{(1)}^{-1} \otimes B_{(2)}^{-1}) (\chi_\alpha^{\eta_1, \eta_2})^* = \eta_2 \chi_\alpha^{\eta_1, \eta_2} \quad (3.6)$$

for all values of η_1, η_2, α . Since this decomposition is now canonical in terms of the degeneracy index α , we will no longer indicate it explicitly.

3.2 Decomposing onto Killing spinors

An arbitrary 32-component complex spinor ε may be decomposed onto the above Killing spinors as follows,

$$\varepsilon = \sum_{\eta_1, \eta_2 = \pm} \chi^{\eta_1, \eta_2} \otimes \zeta_{\eta_1, \eta_2} \quad (3.7)$$

where ζ_{η_1, η_2} is a complex 2-component spinor for each η_1, η_2 , and the 4-fold degeneracy has not been indicated explicitly. As a supersymmetry generator in Type IIB, the spinor ε must be of definite chirality $\Gamma^{11}\varepsilon = -\varepsilon$, which places the following chirality requirements on ζ ,

$$\gamma_{(3)}\zeta_{-\eta_1, -\eta_2} = -\zeta_{\eta_1, \eta_2} \quad (3.8)$$

The charge conjugate spinor is given by,

$$\mathcal{B}^{-1}\varepsilon^* = iB_{(1)}^{-1} \otimes \gamma_{(2)}B_{(2)}^{-1} \otimes B_{(3)}^{-1} \sum_{\eta_1, \eta_2} (\chi^{\eta_1, \eta_2})^* \otimes \zeta_{\eta_1, \eta_2}^* \quad (3.9)$$

Since $\gamma_{(2)}$ anticommutes with $B_{(2)}$, we obtain,

$$\mathcal{B}^{-1}\varepsilon^* = iI_8 \otimes \gamma_{(2)} \otimes I_2 \sum_{\eta_1, \eta_2} \left(B_{(1)}^{-1} \otimes B_{(2)}^{-1} (\chi^{\eta_1, \eta_2})^* \right) \otimes \left(B_{(3)}^{-1} \zeta_{\eta_1, \eta_2}^* \right) \quad (3.10)$$

It will be convenient to denote the result as follows,

$$\mathcal{B}^{-1}\varepsilon^* = \sum_{\eta_1, \eta_2} \chi^{\eta_1, \eta_2} \otimes \star \zeta_{\eta_1, \eta_2} \quad \star \zeta_{\eta_1, \eta_2} = -i\eta_2 \sigma^2 \zeta_{\eta_1, -\eta_2}^* \quad (3.11)$$

As in [19] we will use the τ matrix notation introduced originally in [29] in order to compactly express the action of the various γ matrices on ζ . Defining $\tau^{(ij)} = \tau^i \otimes \tau^j$ with $i, j = 0, 1, 2, 3$, τ^0 the identity matrix and τ^i with $i = 1, 2, 3$ the standard Pauli matrices, we can write,

$$(\tau^{(ij)}\zeta)_{\eta_1, \eta_2} \equiv \sum_{\eta'_1, \eta'_2} (\tau^i)_{\eta_1 \eta'_1} (\tau^j)_{\eta_2 \eta'_2} \zeta_{\eta'_1 \eta'_2} \quad (3.12)$$

3.3 Symmetries of the reduced BPS equations

Using the decomposition of ε into Killing spinors and the streamlined notation of the τ matrices we can finally write down the BPS equations in a reduced form. The reduced dilatino equation is,

$$0 = -4p_a \gamma^a \sigma^2 \zeta^* + g_a \tau^{(03)} \gamma^a \zeta \quad (3.13)$$

while the reduced gravitino equations take the following form,

$$\begin{aligned} (m) \quad 0 &= -\frac{i}{2f_6} \tau^{(21)} \zeta + \frac{D_a f_6}{2f_6} \gamma^a \zeta - \frac{1}{16} g_a \tau^{(03)} \gamma^a \sigma^2 \zeta^* \\ (i) \quad 0 &= \frac{1}{2f_2} \tau^{(02)} \zeta + \frac{D_a f_2}{2f_2} \gamma^a \zeta + \frac{3}{16} g_a \tau^{(03)} \gamma^a \sigma^2 \zeta^* \\ (a) \quad 0 &= \left(D_a + \frac{i}{2} \hat{\omega}_a \sigma^3 - \frac{i}{2} q_a \right) \zeta + \frac{3}{16} g_a \tau^{(03)} \sigma^2 \zeta^* - \frac{1}{16} g_b \tau^{(03)} \gamma_a{}^b \sigma^2 \zeta^* \end{aligned} \quad (3.14)$$

The derivative D_a is defined with respect to the frame e^a , so that the total differential d takes the form $d = e^a D_a$, while the U(1)-connection with respect to frame indices is $\hat{\omega}_a$.

Before we move on to solving these equations, let us briefly look at their symmetries. The axion/dilaton field B transforms non-linearly under SU(1, 1) of Type IIB supergravity and takes values in the coset SU(1, 1)/U(1)_q. Global SU(1, 1) transformations on the fields

are accompanied by local $U(1)_q$ gauge transformations, given in (2.9), and which induce the following symmetry transformations on the fields of the reduced BPS equations,

$$\begin{aligned} U(1)_q \quad \zeta &\rightarrow e^{i\theta/2} \zeta \\ q_a &\rightarrow q_a + D_a \theta \\ p_a &\rightarrow e^{2i\theta} p_a \\ g_a &\rightarrow e^{i\theta} g_a \end{aligned} \quad (3.15)$$

The reduced BPS equations are also invariant under the following discrete symmetries,

$$\begin{aligned} \zeta &\rightarrow \mathcal{I} \zeta = -\tau^{(11)} \sigma^3 \zeta \\ \zeta &\rightarrow \mathcal{J} \zeta = \tau^{(32)} \zeta \end{aligned} \quad (3.16)$$

which leave all the bosonic fields invariant. Both \mathcal{I} and \mathcal{J} commute with $U(1)_q$. Finally, complex conjugation is naturally combined with $U(1)_q$, and we have,

$$\begin{aligned} \zeta &\rightarrow \mathcal{K} \zeta = e^{i\theta} \tau^{(22)} \sigma^2 \zeta^* \\ q_a &\rightarrow \mathcal{K} q_a = -q_a + 2D_a \theta \\ p_a &\rightarrow \mathcal{K} p_a = e^{4i\theta} \bar{p}_a \\ g_a &\rightarrow \mathcal{K} g_a = -e^{2i\theta} \bar{g}_a \end{aligned} \quad (3.17)$$

The chirality requirement of Type IIB restricts the spinor ζ to the subspace,

$$\mathcal{I} \zeta = -\tau^{(11)} \sigma^3 \zeta = \zeta \quad (3.18)$$

In the next subsection, we will investigate the restrictions to the eigen-spaces of \mathcal{J} and \mathcal{K} imposed by the reduced BPS equations. The symmetries $\mathcal{I}, \mathcal{J}, \mathcal{K}$ commute with one another, so that we may diagonalize them simultaneously, and restrict to any one of their common subspaces.

3.4 Restricting to a single subspace of \mathcal{J}

We will assume that p_a does not vanish identically. We now use the dilatino equation to derive a first set of bilinear relations. Multiply the dilatino equation to the left by $\zeta^t T \sigma^2$ and choose the τ -matrix T so that the flux part vanishes,

$$g_a \zeta^t T \tau^{(03)} \sigma^2 \gamma^a \zeta = 0 \quad (3.19)$$

Since $\sigma^2 \gamma^a$ is symmetric for $a = 8, 9$, this condition will require T to satisfy the condition that the product $T \tau^{(03)}$ is anti-symmetric, which has the following solutions,

$$T \in \mathcal{T} = \{\tau^{(01)}, \tau^{(11)}, \tau^{(31)}, \tau^{(20)}, \tau^{(22)}, \tau^{(23)}\} \quad (3.20)$$

The equation implied on p_a is then given by the complex conjugate of,

$$\bar{p}_a \zeta^\dagger T \gamma^a \zeta = 0 \quad (3.21)$$

When $p_a \neq 0$, we can draw from this equation only an orthogonality relation. To obtain a full vanishing condition, we make further use of the chirality condition, and obtain,

$$\bar{p}_a \zeta^\dagger T \tau^{(11)} \gamma^a \sigma^3 \zeta = 0 \quad (3.22)$$

We obtain such a relation when both T and $T \tau^{(11)}$ belong to \mathcal{T} , which is the case for only a single pair, namely $T = \tau^{(20)}, \tau^{(31)}$. As a result, we have the equivalent relations,

$$\zeta^\dagger \tau^{(20)} \gamma^a \zeta = \zeta^\dagger \tau^{(31)} \gamma^a \zeta = 0 \quad (3.23)$$

Next, we analyze the gravitino equations. We multiply equations (m) and (i) of (3.14) on the left by $\zeta^\dagger T \sigma^p$ for $p = 0, 3$, and obtain cancellation of the last term when $T \tau^{(03)}$ is antisymmetric (which is the same condition we had for the dilatino equation),

$$\begin{aligned} 0 &= -\frac{i}{2f_6} \zeta^\dagger T \tau^{(21)} \sigma^p \zeta + \frac{D_a f_6}{2f_6} \zeta^\dagger T \sigma^p \gamma^a \zeta \\ 0 &= \frac{1}{2f_2} \zeta^\dagger T \tau^{(02)} \sigma^p \zeta + \frac{D_a f_2}{2f_2} \zeta^\dagger T \sigma^p \gamma^a \zeta \end{aligned} \quad (3.24)$$

In view of (3.23), the second term will cancel when $T = \tau^{(20)}$ and $T = \tau^{(31)}$, so that we obtain the following relations from the remaining cancellation of the first term,

$$\begin{aligned} \zeta^\dagger \tau^{(01)} \sigma^p \zeta &= 0 & p = 0, 3 \\ \zeta^\dagger \tau^{(22)} \sigma^p \zeta &= 0 \end{aligned} \quad (3.25)$$

and their chiral conjugates, obtained by using the chirality condition $\tau^{(11)} \sigma^3 \zeta = -\zeta$,

$$\begin{aligned} \zeta^\dagger \tau^{(10)} \sigma^p \zeta &= 0 & p = 0, 3 \\ \zeta^\dagger \tau^{(33)} \sigma^p \zeta &= 0 \end{aligned} \quad (3.26)$$

Next, we use the general result of [19] that the bilinear equation $\zeta^\dagger M \zeta = 0$ is solved by projecting ζ onto a subspace with the help of a projection matrix P that anti-commutes with M . This result was established for the case of 2-dimensional M , which is in fact the case also here by reduction. Thus, we must find a projector P which commutes with \mathcal{I} and with the following properties,

$$[P, \tau^{(11)} \sigma^3] = \{P, \tau^{(01)} \sigma^p\} = \{P, \tau^{(22)} \sigma^p\} = 0 \quad (3.27)$$

The solutions to these conditions are $\tau^{(32)}$ and $\tau^{(23)}$, possibly multiplied by a factor of σ^3 . These four possibilities are pairwise equivalent under the chirality relation. Now the projector $P = \tau^{(32)}$ precisely corresponds to the symmetry \mathcal{J} , so imposing a restriction on the spinor space by this operator is the only consistent restriction. Therefore, we will impose,

$$\tau^{(32)} \zeta = \nu \zeta \quad \nu = \pm 1 \quad (3.28)$$

which solves all the above bilinear relations. One must pick one value of ν or the other in the projection.

Imposing the chirality relation (3.18), $\tau^{(11)} \sigma^3 \zeta = -\zeta$, as well as the projector (3.28) we just derived, we may solve for the relations between the components of ζ . Denoting the components by ζ_{abc} , which take values \pm , the a, b components labels the τ -matrix basis, while c labels the chirality basis in which σ^3 is diagonal. We then have two independent complex-valued components which we denote by α and β , and which are defined as follows,

$$\begin{aligned}\bar{\alpha} &= \zeta_{+++} = -\zeta_{--+} = -i\nu\zeta_{+-+} = +i\nu\zeta_{-++} \\ \beta &= \zeta_{---} = +\zeta_{++-} = -i\nu\zeta_{-+-} = -i\nu\zeta_{+--}\end{aligned}\quad (3.29)$$

To reduce the equations to a basis of complex frame indices z, \bar{z} , we will use the following conventions,

$$\begin{aligned}e^z &= \frac{1}{2}(e^8 + ie^9) & \gamma^z &= \frac{1}{2}(\gamma^8 + i\gamma^9) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ e^{\bar{z}} &= \frac{1}{2}(e^8 - ie^9) & \gamma^{\bar{z}} &= \frac{1}{2}(\gamma^8 - i\gamma^9) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\end{aligned}\quad (3.30)$$

In particular, we have,

$$\delta_{z\bar{z}} = 2 \qquad \delta^{z\bar{z}} = \frac{1}{2} \quad (3.31)$$

It will also be convenient to have the following results of $\gamma_a{}^b \sigma^2$,

$$\gamma_z{}^{\bar{z}} \sigma^2 = -\gamma_{\bar{z}}{}^z \sigma^2 = i\sigma^1 \quad (3.32)$$

3.5 The reduced BPS equations in component form

Using the solution (3.29) we found for the projection condition on the preserved supersymmetries, the reduced dilatino equations become,

$$\begin{aligned}4ip_z \alpha - g_z \beta &= 0 \\ 4ip_{\bar{z}} \bar{\beta} + g_{\bar{z}} \bar{\alpha} &= 0\end{aligned}\quad (3.33)$$

The algebraic gravitino equations are,

$$\begin{aligned}\frac{1}{2f_6} \bar{\alpha} + \frac{D_z f_6}{2f_6} \beta - \frac{i}{16} g_z \alpha &= 0 \\ -\frac{1}{2f_6} \beta + \frac{D_{\bar{z}} f_6}{2f_6} \bar{\alpha} + \frac{i}{16} g_{\bar{z}} \bar{\beta} &= 0 \\ \frac{\nu}{2f_2} \bar{\alpha} + \frac{D_z f_2}{2f_2} \beta + \frac{3i}{16} g_z \alpha &= 0 \\ \frac{\nu}{2f_2} \beta + \frac{D_{\bar{z}} f_2}{2f_2} \bar{\alpha} - \frac{3i}{16} g_{\bar{z}} \bar{\beta} &= 0\end{aligned}\quad (3.34)$$

Finally, the component decomposition of the differential equations is as follows,

$$\begin{aligned}
 \left(D_z + \frac{i}{2}\hat{\omega}_z - \frac{i}{2}q_z\right)\bar{\alpha} - \frac{i}{4}g_z\bar{\beta} &= 0 \\
 \left(D_z - \frac{i}{2}\hat{\omega}_z - \frac{i}{2}q_z\right)\beta + \frac{i}{8}g_z\alpha &= 0 \\
 \left(D_{\bar{z}} + \frac{i}{2}\hat{\omega}_{\bar{z}} - \frac{i}{2}q_{\bar{z}}\right)\bar{\alpha} - \frac{i}{8}g_{\bar{z}}\bar{\beta} &= 0 \\
 \left(D_{\bar{z}} - \frac{i}{2}\hat{\omega}_{\bar{z}} - \frac{i}{2}q_{\bar{z}}\right)\beta + \frac{i}{4}g_{\bar{z}}\alpha &= 0
 \end{aligned} \tag{3.35}$$

In addition, we have the complex conjugate equations to all of the equations above. Note that since G and P are complex-valued, we have in general $(g_z)^* \neq g_{\bar{z}}$ and $(p_z)^* \neq p_{\bar{z}}$.

3.6 Determining the radii f_6, f_2 in terms of the spinors

We begin with the two equations that involve $D_z f_6$,

$$\begin{aligned}
 \frac{1}{2f_6}\bar{\alpha} + \frac{D_z f_6}{2f_6}\beta - \frac{i}{16}g_z\alpha &= 0 \\
 -\frac{1}{2f_6}\bar{\beta} + \frac{D_z f_6}{2f_6}\alpha - \frac{i}{16}(g_{\bar{z}})^*\beta &= 0
 \end{aligned} \tag{3.36}$$

To eliminate the contributions from the first term in each equation, we add the first line times $\bar{\beta}$ to the second line times $\bar{\alpha}$, and we obtain,

$$\frac{D_z f_6}{2f_6}(\alpha\bar{\alpha} + \beta\bar{\beta}) = \frac{i}{16}g_z\alpha\bar{\beta} + \frac{i}{16}(g_{\bar{z}})^*\bar{\alpha}\beta \tag{3.37}$$

Proceeding analogously for $D_z f_2$, we have,

$$\begin{aligned}
 \frac{\nu}{2f_2}\bar{\alpha} + \frac{D_z f_2}{2f_2}\beta + \frac{3i}{16}g_z\alpha &= 0 \\
 \frac{\nu}{2f_2}\bar{\beta} + \frac{D_z f_2}{2f_2}\alpha + \frac{3i}{16}(g_{\bar{z}})^*\beta &= 0
 \end{aligned} \tag{3.38}$$

To eliminate the contributions from the first term in each equation, we add the first line times $-\bar{\beta}$ to the second line times $\bar{\alpha}$, and we obtain,

$$\frac{D_z f_2}{2f_2}(\alpha\bar{\alpha} - \beta\bar{\beta}) = \frac{3i}{16}g_z\alpha\bar{\beta} - \frac{3i}{16}(g_{\bar{z}})^*\bar{\alpha}\beta \tag{3.39}$$

These combinations suggest that we should evaluate the covariant derivatives $D_z(\alpha\bar{\alpha} \pm \beta\bar{\beta})$ out of the differential equations (3.35) for $\alpha, \beta, \bar{\alpha}, \bar{\beta}$, and we find,

$$\begin{aligned}
 D_z(\alpha\bar{\alpha} + \beta\bar{\beta}) &= \frac{i}{8}g_z\alpha\bar{\beta} + \frac{i}{8}(g_{\bar{z}})^*\bar{\alpha}\beta \\
 D_z(\alpha\bar{\alpha} - \beta\bar{\beta}) &= \frac{3i}{8}g_z\alpha\bar{\beta} - \frac{3i}{8}(g_{\bar{z}})^*\bar{\alpha}\beta
 \end{aligned} \tag{3.40}$$

We may now eliminate all flux dependences between (3.37), (3.39) and (3.40), and we find,

$$\begin{aligned}\frac{D_z f_6}{f_6}(\alpha\bar{\alpha} + \beta\bar{\beta}) &= D_z(\alpha\bar{\alpha} + \beta\bar{\beta}) \\ \frac{D_z f_2}{f_2}(\alpha\bar{\alpha} - \beta\bar{\beta}) &= D_z(\alpha\bar{\alpha} - \beta\bar{\beta})\end{aligned}\tag{3.41}$$

Given that the arguments of the derivatives are real functions, we conclude,

$$\begin{aligned}f_6 &= c_6 (\alpha\bar{\alpha} + \beta\bar{\beta}) \\ f_2 &= c_2 (\alpha\bar{\alpha} - \beta\bar{\beta})\end{aligned}\tag{3.42}$$

for constant c_2, c_6 . In obtaining the equation for f_2 from the last equation in (3.41), we have assumed that $|\alpha|^2 - |\beta|^2$ does not vanish identically.

3.7 Solving the remaining algebraic gravitino equations

To obtain the results of the previous subsection, we have taken only pairwise linear combinations of the algebraic gravitino equations. Here, we take the orthogonally conjugate pairwise linear combinations, multiplying the first equation by $\bar{\alpha}$ and the second by $-\beta$. The determinant of the two linear combinations is $\alpha\bar{\alpha} + \beta\bar{\beta} \neq 0$, so that the four resulting bilinear equations are guaranteed to be equivalent to the original four algebraic gravitino equations. The terms in $D_z f_6$ and $D_z f_2$ cancel out, and we are left with,

$$\begin{aligned}\frac{1}{2c_6} - \frac{i}{16}g_z\alpha^2 + \frac{i}{16}(g_{\bar{z}})^*\beta^2 &= 0 \\ \frac{\nu}{2c_2} + \frac{3i}{16}g_z\alpha^2 - \frac{3i}{16}(g_{\bar{z}})^*\beta^2 &= 0\end{aligned}\tag{3.43}$$

The last equation may be simplified with the help of the first and yields,

$$c_2 = -\frac{\nu}{3}c_6\tag{3.44}$$

Recall that ν can take the values $\nu = \pm 1$.

3.8 Summary of remaining equations

We may summarize the remaining equations as follows. The dilatino equations are the only ones involving p_z , and may be viewed as determining p_z ,

$$\begin{aligned}4ip_z\alpha - g_z\beta &= 0 \\ 4ip_{\bar{z}}\bar{\beta} + g_{\bar{z}}\bar{\alpha} &= 0\end{aligned}\tag{3.45}$$

Next, we have the radii in terms of the spinors,

$$\begin{aligned}f_6 &= c_6 (\alpha\bar{\alpha} + \beta\bar{\beta}) \\ f_2 &= -\frac{\nu}{3}c_6 (\alpha\bar{\alpha} - \beta\bar{\beta})\end{aligned}\tag{3.46}$$

and the algebraic relation between the spinors and the fluxes,

$$\frac{1}{2c_6} - \frac{i}{16}g_z\alpha^2 + \frac{i}{16}(g_{\bar{z}})^*\beta^2 = 0 \quad (3.47)$$

and finally the differential equations,

$$\begin{aligned} \left(D_z - \frac{i}{2}\hat{\omega}_z + \frac{i}{2}q_z\right)\alpha + \frac{i}{8}(g_{\bar{z}})^*\beta &= 0 \\ \left(D_z - \frac{i}{2}\hat{\omega}_z - \frac{i}{2}q_z\right)\beta + \frac{i}{8}g_z\alpha &= 0 \\ \left(D_z + \frac{i}{2}\hat{\omega}_z - \frac{i}{2}q_z\right)\bar{\alpha} - \frac{i}{4}g_z\bar{\beta} &= 0 \\ \left(D_z + \frac{i}{2}\hat{\omega}_z + \frac{i}{2}q_z\right)\bar{\beta} - \frac{i}{4}(g_{\bar{z}})^*\bar{\alpha} &= 0 \end{aligned} \quad (3.48)$$

along with their complex conjugates.

4 Local solutions to the BPS equations

In the previous section the BPS equations were reduced and solved for the radii f_6, f_2 . The remaining equations are all complex-valued and are organized as follows: we have one algebraic relation (3.47) and four differential equations (3.48) for the spinor components α, β , the reduced flux fields g_z, g_z^* and their complex conjugates. Two further algebraic equations give the axion-dilaton field p_a in terms of the spinors and the fluxes. In this section we will solve completely for the local solutions to this system. Specifically, we will derive expressions for all supergravity fields that satisfy the BPS equations, the Bianchi identities, and the supergravity field equations, in terms of two locally holomorphic functions $\mathcal{A}_+, \mathcal{A}_-$ on Σ . We will discuss the conditions on the local solutions imposed by the proper Minkowski signature of the metric and the absence of singularities in section 5.

The construction of the local solution is involved and proceeds in a number of steps which we will now outline, and carry out in this section in detail. First, we will eliminate the reduced flux fields g_z and $g_{\bar{z}}$ in favor of the reduced axion-dilaton fields $p_z, p_{\bar{z}}$ and similarly for their complex conjugates. Second, we will use the expression for the reduced axion-dilaton fields $p_z, p_{\bar{z}}, q_z, q_{\bar{z}}$ in terms of ρ and B to decouple and integrate one pair of the differential equations, and obtain the spinor components $\bar{\alpha}, \bar{\beta}$ in terms of two holomorphic one-forms κ_{\pm} as well as B and ρ . Third, we will eliminate the spinor components α, β in favor of κ_{\pm}, ρ , and B as well, and thereby produce three nonlinear partial differential equations for the complex field B and the real field ρ . Being non-linear, these equations are not easy to solve, and give rise to a situation reminiscent of [19]. However, in a fourth step we will identify a sequence of two changes of variables which decouples these non-linear differential equations. In a final fifth step, we will solve all decoupled equations in terms of two holomorphic functions \mathcal{A}_{\pm} which are related to κ_{\pm} by $\kappa_{\pm} = \partial_w \mathcal{A}_{\pm}$. For the reader who wishes to skip this entire derivation we have summarized the final result in the introduction to section 5.2.

4.1 Eliminating the reduced flux fields (step 1/5)

First, we eliminate the reduced flux fields $g_z, g_{\bar{z}}$ and their complex conjugates in favor of $p_z, p_{\bar{z}}$ and their complex conjugates, using the dilatino BPS equations (3.45),

$$g_z = 4ip_z \frac{\alpha}{\beta} \quad (g_{\bar{z}})^* = 4i(p_{\bar{z}})^* \frac{\beta}{\alpha} \quad (4.1)$$

The algebraic relation (3.47) becomes,

$$p_z \frac{\alpha^3}{\beta} - (p_{\bar{z}})^* \frac{\beta^3}{\alpha} + \frac{2}{c_6} = 0 \quad (4.2)$$

The differential equations (3.48) take the following form,

$$\begin{aligned} \left(D_z - \frac{i}{2} \hat{\omega}_z + \frac{i}{2} q_z \right) \alpha - \frac{1}{2} (p_{\bar{z}})^* \frac{\beta^2}{\alpha} &= 0 \\ \left(D_z - \frac{i}{2} \hat{\omega}_z - \frac{i}{2} q_z \right) \beta - \frac{1}{2} p_z \frac{\alpha^2}{\beta} &= 0 \\ \left(D_z + \frac{i}{2} \hat{\omega}_z - \frac{i}{2} q_z \right) \bar{\alpha} + p_z \frac{\alpha \bar{\beta}}{\beta} &= 0 \\ \left(D_z + \frac{i}{2} \hat{\omega}_z + \frac{i}{2} q_z \right) \bar{\beta} + (p_{\bar{z}})^* \frac{\bar{\alpha} \beta}{\alpha} &= 0 \end{aligned} \quad (4.3)$$

Equations (4.2) and (4.3) are the remaining relations to be solved. The integrability conditions on the differential equations reproduce the Bianchi identities for the fields P and Q .

4.2 Integrating the first pair of differential equations (step 2/5)

In the second step, we show that the first two equations of (4.3) can be solved in terms of holomorphic functions. Multiplying the first equation of (4.3) by α and the second equation of (4.3) by β , we get equivalently,

$$\begin{aligned} (D_z - i\hat{\omega}_z + iq_z) \alpha^2 - (p_{\bar{z}})^* \beta^2 &= 0 \\ (D_z - i\hat{\omega}_z - iq_z) \beta^2 - p_z \alpha^2 &= 0 \end{aligned} \quad (4.4)$$

We switch to conformally flat complex coordinates (w, \bar{w}) on Σ , such that the metric reads $ds_\Sigma^2 = 4\rho^2 dw d\bar{w}$ and we have,

$$\begin{aligned} e^z &= \rho dw & D_z &= \rho^{-1} \partial_w & \hat{\omega}_z &= i\rho^{-2} \partial_w \rho \\ e^{\bar{z}} &= \rho d\bar{w} & D_{\bar{z}} &= \rho^{-1} \partial_{\bar{w}} & \hat{\omega}_{\bar{z}} &= -i\rho^{-2} \partial_{\bar{w}} \rho \end{aligned} \quad (4.5)$$

The extra factor of ρ^{-1} in the derivatives $D_z, D_{\bar{z}}$ is due to z, \bar{z} being frame indices. We then express p_z and q_z in terms of the complex field B using (2.5), to recast (4.4) as follows,

$$\begin{aligned} \partial_w(\rho \alpha^2) &= -\frac{1}{2} f^2 (B \partial_w \bar{B} - \bar{B} \partial_w B) \rho \alpha^2 + f^2 (\partial_w \bar{B}) \rho \beta^2 \\ \partial_w(\rho \beta^2) &= +\frac{1}{2} f^2 (B \partial_w \bar{B} - \bar{B} \partial_w B) \rho \beta^2 + f^2 (\partial_w B) \rho \alpha^2 \end{aligned} \quad (4.6)$$

By taking suitable linear combinations it is straightforward to verify that the following two equations are equivalent to (4.6),

$$\begin{aligned}\partial_w \left(\ln \{ \rho (\alpha^2 - \bar{B} \beta^2) \} + \ln f \right) &= 0 \\ \partial_w \left(\ln \{ \rho (B \alpha^2 - \beta^2) \} + \ln f \right) &= 0\end{aligned}\quad (4.7)$$

These equations are solved in terms of two independent holomorphic 1-forms κ_{\pm} , as follows,

$$\begin{aligned}\rho f (\alpha^2 - \bar{B} \beta^2) &= \bar{\kappa}_+ \\ \rho f (\beta^2 - B \alpha^2) &= \bar{\kappa}_-\end{aligned}\quad (4.8)$$

Inverting the relation (4.8), we get expressions for the spinor components α, β , and their complex conjugates $\bar{\alpha}, \bar{\beta}$,

$$\begin{aligned}\rho \alpha^2 &= f(\bar{\kappa}_+ + \bar{B} \bar{\kappa}_-) & \rho \bar{\alpha}^2 &= f(\kappa_+ + B \kappa_-) \\ \rho \beta^2 &= f(B \bar{\kappa}_+ + \bar{\kappa}_-) & \rho \bar{\beta}^2 &= f(\bar{B} \kappa_+ + \kappa_-)\end{aligned}\quad (4.9)$$

The right side of all four equations involves only the holomorphic data κ_{\pm} and the B -field and their complex conjugates. It remains to solve for the ρ and B -fields.

4.3 Solving the second pair of differential equations (step 3/5)

In the third step we express the third and fourth equation of (4.3) in terms of B, ρ and the local complex coordinates giving,

$$\begin{aligned}\left(\partial_w - 2(\partial_w \ln \rho) - f^2 B \partial_w \bar{B} \right) (f \rho \bar{\alpha}^2) + 2f^2 (\partial_w B) f \rho \frac{\alpha \bar{\alpha} \bar{\beta}}{\beta} &= 0 \\ \left(\partial_w - 2(\partial_w \ln \rho) - f^2 \bar{B} \partial_w B \right) (f \rho \bar{\beta}^2) + 2f^2 (\partial_w \bar{B}) f \rho \frac{\bar{\alpha} \beta \bar{\beta}}{\alpha} &= 0\end{aligned}\quad (4.10)$$

Next, we compute the derivatives of $\bar{\alpha}$ and $\bar{\beta}$ using (4.3). After taking suitable linear combinations, with coefficients $(\bar{B}, 1)$ and $(1, B)$ of the resulting equations, we find the following equivalent system,

$$\begin{aligned}-(\partial_w \bar{B}) f \rho \bar{\alpha}^2 - 2(\partial_w \bar{B}) \rho f \frac{\bar{\alpha} \beta \bar{\beta}}{\alpha} + \frac{2}{f^2} (\partial_w \ln \rho) f \rho \bar{\beta}^2 &= \bar{B} \partial_w \kappa_+ + \partial_w \kappa_- \\ -(\partial_w B) f \rho \bar{\beta}^2 - 2(\partial_w B) \rho f \frac{\alpha \bar{\alpha} \bar{\beta}}{\beta} + \frac{2}{f^2} (\partial_w \ln \rho) f \rho \bar{\alpha}^2 &= \partial_w \kappa_+ + B \partial_w \kappa_-\end{aligned}\quad (4.11)$$

One now eliminates the spinor components using (4.9) in terms of B, f and ρ , and we obtain, after some simplifications,

$$\begin{aligned}2\partial_w \ln \rho - f^2 (\partial_w \bar{B}) \frac{\kappa_+ + B \kappa_-}{\bar{B} \kappa_+ + \kappa_-} - 2f^2 (\partial_w \bar{B}) e^{+i\vartheta} &= \frac{\bar{B} \partial_w \kappa_+ + \partial_w \kappa_-}{\bar{B} \kappa_+ + \kappa_-} \\ 2\partial_w \ln \rho - f^2 (\partial_w B) \frac{\bar{B} \kappa_+ + \kappa_-}{\kappa_+ + B \kappa_-} - 2f^2 (\partial_w B) e^{-i\vartheta} &= \frac{\partial_w \kappa_+ + B \partial_w \kappa_-}{\kappa_+ + B \kappa_-}\end{aligned}\quad (4.12)$$

Here we have used the following abbreviation for the phase angle ϑ ,

$$e^{2i\vartheta} = \left(\frac{\kappa_+ + B\kappa_-}{\bar{\kappa}_+ + \bar{B}\bar{\kappa}_-} \right) \left(\frac{B\bar{\kappa}_+ + \bar{\kappa}_-}{\bar{B}\kappa_+ + \kappa_-} \right) \quad (4.13)$$

Note that by subtracting the two equations (4.12) the dependence on the metric factor ρ can be eliminated and one arrives at an equation for B and \bar{B} only in terms of κ_{\pm} ,

$$\begin{aligned} & f^4(\partial_w B)(\bar{B}\kappa_+ + \kappa_-)^2 - f^4(\partial_w \bar{B})(\kappa_+ + B\kappa_-)^2 \\ & + 2f^4(\kappa_+ + B\kappa_-)(\bar{B}\kappa_+ + \kappa_-) \left((\partial_w B) e^{-i\vartheta} - (\partial_w \bar{B}) e^{+i\vartheta} \right) \\ & = \kappa_+ \partial_w \kappa_- - \kappa_- \partial_w \kappa_+ \end{aligned} \quad (4.14)$$

The equations (4.12) are of course supplemented by their complex conjugates. While eq. (4.14) seemingly depends on both holomorphic one-forms κ_+ and κ_- , in fact it depends only on their ratio. Assuming that κ_- does not vanish identically, we define the holomorphic or meromorphic function λ by,

$$\lambda = \frac{\kappa_+}{\kappa_-} \quad (4.15)$$

it is straightforward to show that (4.14) depends on λ alone and takes the form,

$$\begin{aligned} & f^4(\partial_w B)(\bar{B}\lambda + 1)^2 - f^4(\partial_w \bar{B})(\lambda + B)^2 \\ & + 2f^4(\lambda + B)(\bar{B}\lambda + 1) \left((\partial_w B) e^{-i\vartheta} - (\partial_w \bar{B}) e^{i\vartheta} \right) = -\partial_w \lambda \end{aligned} \quad (4.16)$$

where the phase of (4.13) is now given by the expression,

$$e^{2i\vartheta} = \frac{(\lambda + B)(\bar{B}\lambda + 1)}{(\bar{\lambda} + \bar{B})(\bar{B}\lambda + 1)} \quad (4.17)$$

The last equation we have to deal with is the algebraic relation (4.2). As before, it may be expressed in terms of B, ρ and the local complex coordinates giving,

$$f^2(\partial_w B) \frac{\rho \alpha^3}{\beta} - f^2(\partial_w \bar{B}) \frac{\rho \beta^3}{\alpha} + \frac{2\rho^2}{c_6} = 0 \quad (4.18)$$

Eliminating $\rho \alpha^2$ and $\rho \beta^2$ using (4.9), we obtain,

$$f^3(\partial_w B) \frac{(\bar{\kappa}_+ + \bar{B}\bar{\kappa}_-)^{\frac{3}{2}}}{(B\bar{\kappa}_+ + \bar{\kappa}_-)^{\frac{1}{2}}} - f^3(\partial_w \bar{B}) \frac{(B\bar{\kappa}_+ + \bar{\kappa}_-)^{\frac{3}{2}}}{(\bar{\kappa}_+ + \bar{B}\bar{\kappa}_-)^{\frac{1}{2}}} + \frac{2\rho^2}{c_6} = 0 \quad (4.19)$$

Its dependence on κ_{\pm} cannot be reduced to a dependence solely on λ since the one-form $\bar{\kappa}_-$ is needed to combine with ρ^2 to produce an equation in which all terms transform as $(1, 0)$ forms, and we obtain,

$$f^3(\partial_w B) \frac{(\bar{\lambda} + \bar{B})^{\frac{3}{2}}}{(B\bar{\lambda} + 1)^{\frac{1}{2}}} - f^3(\partial_w \bar{B}) \frac{(B\bar{\lambda} + 1)^{\frac{3}{2}}}{(\bar{\lambda} + \bar{B})^{\frac{1}{2}}} + \frac{2\rho^2}{c_6 \bar{\kappa}_-} = 0 \quad (4.20)$$

In summary, we have reduced the remaining BPS equations and expressed them in terms of complex differential equations given by (4.12) and (4.19) along with their complex conjugate equations. We will show in appendix D that these equations imply that the Bianchi identities are satisfied, and that there are hence no more constraints to take into account. Integrating these equations will give the complex scalar field B and the metric field ρ in terms of the holomorphic one forms κ_{\pm} . In the next section we will perform several variable redefinitions which bring the equations into a form where they can be decoupled and integrated.

4.4 Decoupling by changing variables (step 4/5)

In this subsection, we will perform two consecutive changes of variables to decouple the remaining equations. The corresponding choices will be motivated first and then carried out on the equations.

4.4.1 First change of variables, from B to Z

A first change of variables replaces B by a complex field Z and is designed to parametrize the phase $e^{i\vartheta}$ in (4.16) without the square root required from its definition in (4.17). The following rational change of variables eliminates B in terms of a complex function Z by,

$$Z^2 = \frac{\lambda + B}{B\bar{\lambda} + 1} \quad B = \frac{Z^2 - \lambda}{1 - \bar{\lambda}Z^2} \quad (4.21)$$

and will allow us to express $e^{i\vartheta}$ and f^2 as rational functions of Z and its complex conjugate,

$$e^{i\vartheta} = \frac{Z}{\bar{Z}} \left(\frac{1 - \lambda\bar{Z}^2}{1 - \bar{\lambda}Z^2} \right) \quad f^2 = \frac{(1 - \lambda\bar{Z}^2)(1 - \bar{\lambda}Z^2)}{(1 - |\lambda|^2)(1 - |Z|^4)} \quad (4.22)$$

The derivatives of B and \bar{B} take the following form,

$$\begin{aligned} \partial_w B &= \frac{1 - |\lambda|^2}{(1 - \bar{\lambda}Z^2)^2} \partial_w Z^2 - \frac{\partial_w \lambda}{1 - \bar{\lambda}Z^2} \\ \partial_w \bar{B} &= \frac{1 - |\lambda|^2}{(1 - \lambda\bar{Z}^2)^2} \partial_w \bar{Z}^2 + \frac{\bar{Z}^2(\bar{Z}^2 - \bar{\lambda})\partial_w \lambda}{(1 - \lambda\bar{Z}^2)^2} \end{aligned} \quad (4.23)$$

Implementing this change of variables on equation (4.16) produces the following form,

$$\begin{aligned} (2 + 4|Z|^2)\partial_w Z - Z^2(4 + 2|Z|^2)\partial_w \bar{Z} \\ = \frac{2\bar{Z}(1 + |Z|^2 + |Z|^4) - \bar{\lambda}Z(1 + 4|Z|^2 + |Z|^4)}{1 - |\lambda|^2} \partial_w \lambda \end{aligned} \quad (4.24)$$

Equation (4.16) was just one linear combination of the two equations in (4.12), given by the difference of the two equations (4.12). We will take the first equation of (4.12) as the complimentary independent equation, and eliminate its B -dependence in favor of Z ,

$$\partial_w \ln \left(\frac{\rho^2}{\kappa - \bar{\kappa}} \frac{|1 - Z^2\bar{\lambda}|(1 - |Z|^2)}{f|Z|(1 - |\lambda|^2)(1 + |Z|^2)} \right) = \frac{1}{2} \partial_w \ln \frac{Z}{\bar{Z}} - \frac{\bar{Z}}{Z} \left(\frac{\partial_w \lambda}{1 - |\lambda|^2} \right) \quad (4.25)$$

Finally, eliminating B in favor of Z in the algebraic flux equation (4.20) as well, we obtain,

$$(1 - |\lambda|^2)\partial_w \left(\frac{Z^2 + \bar{Z}^{-2}}{1 - |\lambda|^2} \right) - \frac{2\partial_w \lambda}{1 - |\lambda|^2} + 2\hat{\rho}^2 \kappa_- \frac{|Z|}{\bar{Z}^3} (1 + |Z|^2)^2 = 0 \quad (4.26)$$

It remains to solve the system of equations (4.24), (4.25) and (4.26).

4.4.2 Second change of variables, from Z to R, ψ

A second change of variables is inspired by the form of equation (4.25), in which the norm of Z and its phase enter in distinct parts of the equation. We express the complex field Z in terms of two real variables, the absolute value R of Z , and its phase ψ ,

$$Z^2 = R e^{i\psi} \quad (4.27)$$

It will also be useful to change variables from ρ to $\hat{\rho}$ in the following way,

$$\hat{\rho}^2 = \frac{\rho^2}{\kappa_- \bar{\kappa}_-} \frac{|1 - Z^2 \bar{\lambda}|(1 - |Z|^2)}{f|Z|(1 - |\lambda|^2)(1 + |Z|^2)} \quad (4.28)$$

In terms of these variables (4.25) takes the form,

$$\partial_w \ln \hat{\rho}^2 - \frac{i}{2} \partial_w \psi + e^{-i\psi} \frac{\partial_w \lambda}{1 - |\lambda|^2} = 0 \quad (4.29)$$

while (4.24) becomes,

$$(1 - R^2) \frac{\partial_w R}{R} + (1 + 4R + R^2) \left(i \partial_w \psi + \frac{\bar{\lambda} \partial_w \lambda}{1 - |\lambda|^2} \right) - \frac{2e^{-i\psi}(1 + R + R^2)}{1 - |\lambda|^2} \partial_w \lambda = 0 \quad (4.30)$$

and (4.26) becomes,

$$\left(R - \frac{1}{R} \right) \partial_w R + (R^2 + 1) \left(i \partial_w \psi + \frac{\bar{\lambda} \partial_w \lambda}{1 - |\lambda|^2} \right) - \frac{2R \partial_w \lambda}{1 - |\lambda|^2} e^{-i\psi} + 2\hat{\rho}^2 \kappa_- e^{i\psi/2} (1 + R)^2 = 0 \quad (4.31)$$

The three equations (4.29), (4.30), and (4.31) are the basic starting point for the complete solution of the full system of reduced BPS equations.

4.4.3 Decoupling the equations for ψ and $\hat{\rho}^2$

Adding (4.30) and (4.31) cancels the terms proportional to $\partial_w R$. Remarkably, and the secret to decoupling the equations, is now that the entire R -dependence of this sum resides in an overall multiplicative factor of $(1 + R)^2$. Omitting this factor, the sum becomes,

$$2i \partial_w \psi + \frac{2\bar{\lambda} \partial_w \lambda}{1 - |\lambda|^2} - \frac{2\partial_w \lambda}{1 - |\lambda|^2} e^{-i\psi} + 2\hat{\rho}^2 \kappa_- e^{i\psi/2} = 0 \quad (4.32)$$

Together with (4.29), which we repeat here for convenience,

$$\partial_w \ln \hat{\rho}^2 - \frac{i}{2} \partial_w \psi + \frac{\partial_w \lambda}{1 - |\lambda|^2} e^{-i\psi} = 0 \quad (4.33)$$

equation (4.32) forms a system of equations for only two of the three real fields of the system, namely ψ and $\hat{\rho}^2$. This system is very similar to the one solved in [19], and we will approach it with similar methods. We note in passing that the integrability condition on (4.33) viewed as an equation for $\hat{\rho}^2$ as a function of ψ and given λ is given by,

$$2\partial_w \partial_{\bar{w}} \psi + \partial_{\bar{w}} \left(\frac{2i \partial_w \lambda}{1 - |\lambda|^2} e^{-i\psi} \right) - \partial_w \left(\frac{2i \partial_{\bar{w}} \bar{\lambda}}{1 - |\lambda|^2} e^{+i\psi} \right) = 0 \quad (4.34)$$

which is a conformal invariant field equation for ψ of the sine-Gordon Liouville type [35], very similar to equation (1.3) of [19]. The equation for R will be dealt with in section 4.5.2.

Adding twice (4.33) to (4.32) eliminates the term proportional to $e^{-i\psi}$, and we obtain,

$$\partial_w \ln \hat{\rho}^2 + \frac{i}{2} \partial_w \psi - \partial_w \ln(1 - |\lambda|^2) + \hat{\rho}^2 \kappa_- e^{i\psi/2} = 0 \quad (4.35)$$

Clearly, this equation involves only the following specific complex combination of $\hat{\rho}^2$ and ψ ,

$$\xi = \frac{1}{\hat{\rho}^2 e^{i\psi/2}} \quad (4.36)$$

in terms of which we may express (4.35) as follows,

$$\partial_w \left(\xi(1 - |\lambda|^2) \right) = \kappa_- (1 - |\lambda|^2) = \kappa_- - \kappa_+ \bar{\lambda} \quad (4.37)$$

where we have used the relation $\kappa_+ = \lambda \kappa_-$.

With the help of this sequence of changes of variables, the integrable structure of the system of equations (4.24), (4.25) and (4.26) has been brought out clearly. Indeed, equation (4.37) involves only the field ξ , which is the particular combination of $\hat{\rho}$ and ψ used in (4.36). Having solved for ξ , either equation (4.33) or (4.35) becomes an equation for only a single variable, either $\hat{\rho}$ or ψ , and may be solved. Finally, having $\hat{\rho}$ and ψ , equation (4.30) becomes an equation for R only, and we will see below that it may be solved as well.

4.5 Solving for ψ , $\hat{\rho}^2$ and R in terms of \mathcal{A}_\pm (step 5/5)

Having decoupled the reduced BPS equations in the preceding subsection, we will solve the decoupled equations in the present section. To do so, we begin by solving equation (4.37) for ξ , and then obtain ψ , $\hat{\rho}^2$ and R . Introducing locally holomorphic functions \mathcal{A}_\pm such that,

$$\partial_w \mathcal{A}_\pm = \kappa_\pm \quad (4.38)$$

the function λ may be expressed in terms of \mathcal{A}_\pm by,

$$\lambda = \frac{\partial_w \mathcal{A}_+}{\partial_w \mathcal{A}_-} \quad (4.39)$$

Given the one-forms κ_\pm , the functions \mathcal{A}_\pm are unique up to an additive constant for each function. Viewed as equations for the supergravity fields in terms of \mathcal{A}_\pm , the reduced BPS equations are therefore invariant under shifting the holomorphic functions \mathcal{A}_\pm by arbitrary complex constants.

4.5.1 Solving for ξ

In terms of \mathcal{A}_\pm equation (4.37) may be recast in the following form,

$$\partial_w \left(\xi(1 - \lambda \bar{\lambda}) - \mathcal{A}_- + \mathcal{A}_+ \bar{\lambda} \right) = 0 \quad (4.40)$$

Equation (4.40) is solved in terms of an arbitrary locally holomorphic function \mathcal{A}_0 by,

$$\xi(1 - \lambda \bar{\lambda}) = \mathcal{A}_- - \mathcal{A}_+ \bar{\lambda} + \bar{\mathcal{A}}_0 \quad (4.41)$$

which provides the general solution to (4.37). To determine \mathcal{A}_0 in terms of \mathcal{A}_\pm , we enforce (4.33) on the result (4.41). Upon eliminating ξ using (4.41), we find,

$$(1 - \lambda\bar{\lambda})\partial_w\mathcal{A}_0 = (\partial_w\lambda)\left(\bar{\mathcal{A}}_+ + \bar{\mathcal{A}}_0 + \mathcal{A}_- - \bar{\lambda}(\mathcal{A}_+ + \mathcal{A}_0 + \bar{\mathcal{A}}_-)\right) \quad (4.42)$$

To proceed we change variables from the holomorphic function \mathcal{A}_0 to a new holomorphic function φ , related as follows, $\mathcal{A}_0 = -\mathcal{A}_+ + \lambda(\mathcal{A}_- + \varphi)$. The equation for φ then becomes,

$$(1 - \lambda\bar{\lambda})\partial_w\varphi + \varphi\partial_w\ln\lambda = \bar{\lambda}\bar{\varphi}\partial_w\ln\lambda \quad (4.43)$$

If λ is constant, then φ must be constant as well. Assuming henceforth that $\partial_w\lambda \neq 0$, we take the derivative of the entire equation with respect to \bar{w} , and regroup terms according to their holomorphicity properties,

$$-\frac{\lambda\partial_w\varphi}{\partial_w\ln\lambda} = \frac{\partial_{\bar{w}}(\bar{\lambda}\bar{\varphi})}{\partial_{\bar{w}}\bar{\lambda}} \quad (4.44)$$

The left side is holomorphic, while the right side is anti-holomorphic, and hence both sides must equal a complex constant α , so that we find,

$$\begin{aligned} \lambda\partial_w\varphi &= -\alpha\partial_w\ln\lambda \\ \partial_w(\lambda\varphi) &= \bar{\alpha}\partial_w\lambda \end{aligned} \quad (4.45)$$

Eliminating the derivative $\partial_w\varphi$ between both equations gives,

$$\varphi = \frac{\alpha}{\lambda} + \bar{\alpha} \quad (4.46)$$

Assembling all these results gives,

$$\xi(1 - \lambda\bar{\lambda}) = (\mathcal{A}_- - \bar{\mathcal{A}}_+ + \bar{\alpha}) + \bar{\lambda}(\bar{\mathcal{A}}_- - \mathcal{A}_+ + \alpha) \quad (4.47)$$

Recalling that the functions \mathcal{A}_\pm were defined only up to additive constant shifts, we may absorb the constant α into \mathcal{A}_\pm , so that our final expression for the solution becomes,

$$\xi = \frac{\mathcal{L}}{1 - |\lambda|^2} \quad (4.48)$$

where \mathcal{L} is given by,

$$\mathcal{L} = (\mathcal{A}_- - \bar{\mathcal{A}}_+) + \bar{\lambda}(\bar{\mathcal{A}}_- - \mathcal{A}_+) \quad (4.49)$$

Note that $\hat{\rho}$ and ψ are directly determined by ξ using equation (4.36).

4.5.2 Solving for R

To solve for R , we start from equation (4.30) and eliminate the term proportional to $e^{-i\psi}$ using (4.29). We then divide the resulting equation by R , and find,

$$\begin{aligned} 0 &= \left(\frac{1}{R^2} - 1\right)\partial_w R + \left(R + \frac{1}{R} + 4\right)(i\partial_w\psi - \partial_w\ln(1 - |\lambda|^2)) \\ &\quad + \left(R + \frac{1}{R} + 1\right)(2\partial_w\ln\hat{\rho}^2 - i\partial_w\psi) \end{aligned} \quad (4.50)$$

Changing variables from R to W defined by,

$$W = R + \frac{1}{R} \quad (4.51)$$

renders equation (4.50) linear in W , with an inhomogeneous part,

$$\partial_w W - 2(W + 1)\partial_w \ln \hat{\rho}^2 + (W + 1)\partial_w \ln(1 - \lambda\bar{\lambda}) = 3i\partial_w \psi - 3\partial_w \ln(1 - \lambda\bar{\lambda}) \quad (4.52)$$

The homogeneous equation is solved straightforwardly by,

$$W + 1 = W_0 \frac{\hat{\rho}^4}{1 - \lambda\bar{\lambda}} \quad (4.53)$$

To solve the homogeneous equation, W_0 is an arbitrary constant which is required to be real for real W . To find a particular solution to the inhomogeneous equation, we let W_0 be a function, which then must satisfy,

$$\partial_w W_0 = \frac{1 - \lambda\bar{\lambda}}{\hat{\rho}^4} 3i\partial_w \psi - 3 \frac{\partial_w(1 - \lambda\bar{\lambda})}{\hat{\rho}^4} \quad (4.54)$$

Recasting the equation in terms of the variable $\xi = \hat{\rho}^{-2}e^{i\psi/2}$, and then expressing ξ in terms of \mathcal{L} and λ , using (4.48), we find,

$$\partial_w W_0 = \frac{3\mathcal{L}\bar{\mathcal{L}}}{1 - \lambda\bar{\lambda}} \partial_w \ln \frac{\bar{\mathcal{L}}}{\mathcal{L}} - \frac{3\partial_w(\mathcal{L}\bar{\mathcal{L}})}{1 - \lambda\bar{\lambda}} + \partial_w \left(\frac{3\mathcal{L}\bar{\mathcal{L}}}{1 - \lambda\bar{\lambda}} \right) \quad (4.55)$$

The first two terms on the right side may be evaluated using the expression for \mathcal{L} of (4.49). Putting all together, we find the following equation for W_0 ,

$$\partial_w \left(W_0 - \frac{3\mathcal{L}\bar{\mathcal{L}}}{1 - \lambda\bar{\lambda}} \right) = -6\bar{\mathcal{L}}\partial_w \mathcal{A}_- \quad (4.56)$$

Using the explicit formula for $\bar{\mathcal{L}}$, and the fact that $\lambda\partial_w \mathcal{A}_- = \partial_w \mathcal{A}_+$, we find,

$$\partial_w \left(W_0 - \frac{3\mathcal{L}\bar{\mathcal{L}}}{1 - \lambda\bar{\lambda}} - 6\mathcal{A}_+ \bar{\mathcal{A}}_+ + 6\mathcal{A}_- \bar{\mathcal{A}}_- \right) = 6(\mathcal{A}_+ \partial_w \mathcal{A}_- - \mathcal{A}_- \partial_w \mathcal{A}_+) \quad (4.57)$$

Now the right side is a holomorphic 1-form, and so locally there exists a holomorphic function \mathcal{B} , defined up to the addition of an arbitrary complex constant such that,⁶

$$\mathcal{A}_+ \partial_w \mathcal{A}_- - \mathcal{A}_- \partial_w \mathcal{A}_+ = \partial_w \mathcal{B} \quad (4.58)$$

The general solution is then given by,

$$W_0 = \frac{3\mathcal{L}\bar{\mathcal{L}}}{1 - \lambda\bar{\lambda}} + 6\mathcal{A}_+ \bar{\mathcal{A}}_+ - 6\mathcal{A}_- \bar{\mathcal{A}}_- + 6\mathcal{B} + 6\bar{\mathcal{B}} \quad (4.59)$$

and an arbitrary integration constant, which parametrizes the admixture of the solution to the homogeneous equation, has been absorbed into \mathcal{B} . Note that the entire solution is real, as is required by the nature of W and R . This completes the solution of the decoupled reduced BPS equations for the fields ψ , $\hat{\rho}$ and R .

⁶Clearly, the holomorphic function \mathcal{B} should not be confused with the 10-dimensional charge conjugation matrix of footnote 3 for which the same symbol is being used here.

5 Local solution to type IIB supergravity

In this section we summarize the complete local solution for the supergravity fields, which is parametrized by two holomorphic functions \mathcal{A}_\pm and various constants. The doublet $(\mathcal{A}_+, \mathcal{A}_-)$ transforms linearly under the group $SU(1, 1)$, which is isomorphic to the group $SL(2, \mathbb{R})$. The transformation properties of the supergravity fields under $SU(1, 1)$ can be made transparent by expressing the supergravity fields with the help of natural invariants. The $SU(1, 1)$ transformation properties of the holomorphic data and its invariants will be spelled out in section 5.1, before we give the supergravity fields in 5.2 and discuss the $SU(1, 1)$ transformations induced on them in section 5.3. We discuss positivity and regularity conditions respectively in sections 5.4 and 5.5. In 5.6 we show how the T-dual of the D4/D8 solution in massive type IIA supergravity can be recovered as a special case of our general solution, while in 5.7 we give a local solution which satisfies the positivity and regularity conditions in a finite but local region near a boundary segment. Finally, in section 5.8, we discuss the conditions on the holomorphic data \mathcal{A}_\pm and on the supergravity fields under which solutions with monodromy can exist.

5.1 $SU(1, 1)$ transformations of the holomorphic data

The basic data parametrizing the general local solution are two holomorphic functions \mathcal{A}_\pm , which transform linearly under $SU(1, 1)$. Parametrizing the elements of $SU(1, 1)$ by $u, v \in \mathbb{C}$ subject to $|u|^2 - |v|^2 = 1$, the functions \mathcal{A}_\pm transform as,

$$\begin{aligned}\mathcal{A}_+ &\rightarrow \mathcal{A}'_+ = u\mathcal{A}_+ - v\mathcal{A}_- \\ \mathcal{A}_- &\rightarrow \mathcal{A}'_- = \bar{u}\mathcal{A}_- - \bar{v}\mathcal{A}_+\end{aligned}\tag{5.1}$$

Note that the $\bar{\mathcal{A}}_\mp$ and κ_\pm transform in the same fashion as \mathcal{A}_\pm does. The functions \mathcal{A}_\pm determine the holomorphic function \mathcal{B} introduced in (4.58), up to an additive constant, by,

$$\partial_w \mathcal{B} = \mathcal{A}_+ \partial_w \mathcal{A}_- - \mathcal{A}_- \partial_w \mathcal{A}_+\tag{5.2}$$

The right hand side is invariant under the $SU(1, 1)$ transformations (5.1) and consequently \mathcal{B} transforms at most by a constant shift. Derived quantities which will copiously appear in the expressions for the supergravity fields are κ_\pm and λ defined by,

$$\kappa_\pm = \partial_w \mathcal{A}_\pm \qquad \kappa_+ = \lambda \kappa_-\tag{5.3}$$

Since κ_\pm transform as \mathcal{A}_\pm , we are led to the following natural invariant,

$$\kappa^2 = -|\kappa_+|^2 + |\kappa_-|^2\tag{5.4}$$

Finally, a combination which was already encountered earlier in (4.59) and given by,

$$\mathcal{G} = |\mathcal{A}_+|^2 - |\mathcal{A}_-|^2 + \mathcal{B} + \bar{\mathcal{B}}\tag{5.5}$$

also has simple transformation properties, since the first and second terms on the right side combine into an $SU(1, 1)$ invariant. Since \mathcal{B} transforms at most by shifts under $SU(1, 1)$, so does \mathcal{G} . Moreover, \mathcal{G} is the only place where \mathcal{B} shows up, and we see that of the generally complex constant of integration hiding \mathcal{B} , only the real part is relevant. For later use, we note the following relation between \mathcal{G} and κ^2 given by $\kappa^2 = -\partial_w \partial_{\bar{w}} \mathcal{G}$.

5.2 Supergravity fields in terms of holomorphic data

The general local solution to Type IIB supergravity with $SO(2, 5) \times SO(3)$ symmetry can now be expressed in terms of the holomorphic data introduced above. Translating the local solution to the reduced BPS equations obtained in section 4 back to the supergravity fields is straightforward, except in the case of the flux field for which the derivation of the flux potential is quite involved, and detailed calculations are relegated to appendix E. Here we will summarize the results and discuss some of the immediate properties of the solutions.

Recall that the symmetries of the problem dictate the Ansatz for the bosonic supergravity fields, while the fermionic fields vanish. The five-form field strength vanishes, while the Ansatz for the metric and three-form flux fields are as follows,

$$\begin{aligned} ds^2 &= f_6^2 ds_{\text{AdS}_6}^2 + f_2^2 ds_{\Sigma^2}^2 + ds_{\Sigma}^2 & ds_{\Sigma}^2 &= 4\rho^2 dw d\bar{w} \\ F_{(3)} &= dC_{(2)} & C_{(2)} &= \mathcal{C} \hat{e}^{67} \end{aligned} \quad (5.6)$$

where $f_6, f_2, \rho, \mathcal{C}$ and the dilaton-axion field B are all functions on Σ , and \hat{e}^{67} is the volume form on an S^2 of unit radius defined in (2.15).

The metric functions f_2, f_6 and ρ can be expressed solely in terms of \mathcal{G} , and for notational convenience we introduce the composite quantities R and W defined in terms of \mathcal{G} through,

$$W = R + \frac{1}{R} = 2 + 6 \frac{\kappa^2 \mathcal{G}}{|\partial_w \mathcal{G}|^2} \quad (5.7)$$

The right side is real and so are R and W . The metric functions are then given by,

$$f_2^2 = \frac{c_6^2 \kappa^2 (1 - R)}{9 \rho^2 (1 + R)} \quad f_6^2 = \frac{c_6^2 \kappa^2 (1 + R)}{\rho^2 (1 - R)} \quad \rho^2 = \frac{(R + R^2)^{\frac{1}{2}}}{|\partial_w \mathcal{G}|} \left(\frac{\kappa^2}{1 - R} \right)^{\frac{3}{2}} \quad (5.8)$$

The axion-dilaton field B of (2.7) is given by,

$$B = \frac{\kappa_+ \partial_{\bar{w}} \mathcal{G} - \kappa_- R \partial_w \mathcal{G}}{\bar{\kappa}_+ R \partial_w \mathcal{G} - \bar{\kappa}_- \partial_{\bar{w}} \mathcal{G}} \quad (5.9)$$

Finally, the flux potential function \mathcal{C} for the three-form field strength $F_{(3)}$ is derived in appendix E and given by,

$$\mathcal{C} = \frac{4i c_6^2}{9} \left(\frac{\bar{\kappa}_- W \partial_w \mathcal{G} - 2\kappa_+ \partial_{\bar{w}} \bar{\mathcal{G}}}{(W + 2) \kappa^2} - \bar{\mathcal{A}}_- - 2\mathcal{A}_+ - \mathcal{K}_0 \right) \quad (5.10)$$

Here, \mathcal{K}_0 is a complex integration constant which represents the residual gauge transformation degree of freedom in $C_{(2)}$ restricted to our Ansatz, and does not affect the gauge-invariant field strength $F_{(3)}$.

5.3 $SU(1, 1)$ transformations induced on the supergravity fields

The expressions for the supergravity fields in terms of the holomorphic data, obtained in section 5.2 allow us to specify more precisely which transformations of the holomorphic data leave the supergravity fields invariant, or transform them according to $SU(1, 1)$ -duality.

The radii f_6 and f_2 must be invariant since they parametrize the metric in Einstein frame. Therefore, R and the combination κ^2/ρ^2 must be invariant, and hence $|\partial_w \mathcal{G}|^2/\kappa^2$ and as a result also \mathcal{G} and κ^2 themselves must be invariant by (5.7). We will now implement these invariance requirements on the holomorphic data \mathcal{A}_\pm themselves.

Invariance of κ^2 requires that the holomorphic one-forms transform under $SU(1, 1)$ as,

$$\begin{aligned}\kappa_+ &\rightarrow \kappa'_+ = +u\kappa_+ - v\kappa_- \\ \kappa_- &\rightarrow \kappa'_- = -\bar{v}\kappa_+ + \bar{u}\kappa_-\end{aligned}\tag{5.11}$$

where we have parametrized $SU(1, 1)$ by $u, v \in \mathbb{C}$ with $|u|^2 - |v|^2 = 1$. Integrating the above transformation laws to obtain the holomorphic functions \mathcal{A}_\pm we get,

$$\begin{aligned}\mathcal{A}_+ &\rightarrow \mathcal{A}'_+ = +u\mathcal{A}_+ - v\mathcal{A}_- + a_+ \\ \mathcal{A}_- &\rightarrow \mathcal{A}'_- = -\bar{v}\mathcal{A}_+ + \bar{u}\mathcal{A}_- + a_-\end{aligned}\tag{5.12}$$

where a_\pm are complex constants. The addition of the constants a_\pm leaves κ_\pm unchanged but transforms $\partial_w \mathcal{B}$ as follows,

$$\partial_w \mathcal{B} \rightarrow \partial_w \mathcal{B}' = \partial_w \mathcal{B} + a_+ \partial_w \mathcal{A}'_- - a_- \partial_w \mathcal{A}'_+ \tag{5.13}$$

Integrating this relation, we find,

$$\mathcal{B} \rightarrow \mathcal{B}' = \mathcal{B} + a_+ \mathcal{A}'_- - a_- \mathcal{A}'_+ + b_0 \tag{5.14}$$

where b_0 is a complex constant. Therefore, the transformation law for \mathcal{G} is as follows,

$$\begin{aligned}\mathcal{G} \rightarrow \mathcal{G}' &= \mathcal{G} + (a_+ - \bar{a}_-)(\bar{\mathcal{A}}'_+ + \mathcal{A}'_-) + (\bar{a}_+ - a_-)(\mathcal{A}'_+ + \bar{\mathcal{A}}'_-) \\ &\quad + a_+ \bar{a}_+ - a_- \bar{a}_- + b_0 + \bar{b}_0\end{aligned}\tag{5.15}$$

Invariance of \mathcal{G} requires,

$$a_- = \bar{a}_+ \qquad b_0 + \bar{b}_0 = 0 \tag{5.16}$$

Since \mathcal{A}_- and $\bar{\mathcal{A}}_+$ transform under $SU(1, 1)$ by the same formula, the restriction $a_- = \bar{a}_+$ is automatically $SU(1, 1)$ -invariant. To analyze the transformation properties of the dilaton-axion field B , we use the solution (5.9) and again appeal to the fact that \mathcal{G} , $\partial_w \mathcal{G}$ and R are invariant under $SU(1, 1)$. The transformation property of B under (5.12) is as follows,

$$B \rightarrow B' = \frac{uB + v}{\bar{v}B + \bar{u}} \tag{5.17}$$

Finally, the transformation law for the flux potential $C_{(2)} = \mathcal{C} e^{6\gamma}$ under (5.12) is given by,

$$\mathcal{C} \rightarrow \mathcal{C}' = u\mathcal{C} + v\bar{\mathcal{C}} - \frac{4ic_6^2}{9}(\mathcal{K}'_0 - u\mathcal{K}_0 - v\bar{\mathcal{K}}_0 + 3a_+) \tag{5.18}$$

The last term is constant and amounts to a gauge transformation on the field $C_{(2)}$. Therefore, up to gauge transformations, we recover (2.8), as required. We see that the transformations of (5.1) indeed induce the appropriate $SU(1, 1)$ transformations on the supergravity fields. But at the level of the holomorphic data we have an additional shift symmetry, which leaves the supergravity fields invariant and results from the fact that \mathcal{A}_\pm are determined from κ_\pm only up to additive constants.

5.4 Positivity conditions

By construction in (5.8), the metric fields f_2^2 , f_6^2 and ρ^2 are real, but they are not necessarily positive, as is required by the Minkowski signature of ten-dimensional space-time. In this subsection, we investigate the requirements on the holomorphic data implied by the positivity of f_2^2 , f_6^2 and ρ^2 , and the condition $|B| \leq 1$. There are no reality or positivity requirements derived from the flux field $C_{(2)}$ since it is complex. The further conditions needed to produce regular solutions will be investigated in the next section.

Positivity of the expressions for f_2^2 and f_6^2 in (5.8) requires,

$$\kappa^2(1 - R) \geq 0 \quad (5.19)$$

The expression for ρ^2 in (5.8) is then automatically positive. By definition R is an absolute value, so that we must have $R \geq 0$ and therefore $W \geq 2$ by (5.7), which implies,

$$\kappa^2 \mathcal{G} \geq 0 \quad (5.20)$$

Finally, we verify that $|B| \leq 1$ holds, or equivalently $f^2 \geq 1$, using (4.22),

$$f^2 = \frac{1}{1 - |B|^2} = 1 + \frac{|\lambda - Z^2|^2}{(1 - |\lambda|^2)(1 - |Z|^4)} \quad (5.21)$$

It is manifest that $f^2 \geq 1$ since $\kappa^2(1 - R) = |\kappa_-|^2(1 - |\lambda|^2)(1 - |Z|^2)$, which is positive in view of (5.19). So the conditions (5.19) and (5.20) exhaust the reality constraints. As a result, there are two branches to the solutions,

$$\begin{aligned} \{\kappa^2 \geq 0, \quad R \leq 1, \quad \mathcal{G} \geq 0\} \\ \{\kappa^2 \leq 0, \quad R \geq 1, \quad \mathcal{G} \leq 0\} \end{aligned} \quad (5.22)$$

Actually, these two branches are mapped into one another under a complex conjugation, which includes the reversal of the complex structure on Σ . Specifically, this transformation reverses the sign of κ^2 and \mathcal{G} , maps R to R^{-1} , and interchanges w and \bar{w} , and may be realized on the holomorphic functions as follows,

$$\begin{aligned} \mathcal{A}_+(w) &\rightarrow \mathcal{A}'_+(w) = \bar{\mathcal{A}}_-(w) = \overline{\mathcal{A}_-(\bar{w})} \\ \mathcal{A}_-(w) &\rightarrow \mathcal{A}'_-(w) = \bar{\mathcal{A}}_+(w) = \overline{\mathcal{A}_+(\bar{w})} \end{aligned} \quad (5.23)$$

and thus on holomorphic forms by,

$$\begin{aligned} \kappa_+(w) &\rightarrow \kappa'_+(w) = \bar{\kappa}_-(w) = \overline{\kappa_-(\bar{w})} \\ \kappa_-(w) &\rightarrow \kappa'_-(w) = \bar{\kappa}_+(w) = \overline{\kappa_+(\bar{w})} \end{aligned} \quad (5.24)$$

One verifies that these transformations have the desired action on κ^2 , \mathcal{G} , and R , and therefore leave the metric functions f_2^2, f_6^2, ρ^2 invariant, while complex conjugating the fields B and \mathcal{C} combined with a reversal of the complex structure on Σ ,

$$\begin{aligned} B(w, \bar{w}) &\rightarrow B'(w, \bar{w}) = \bar{B}(w, \bar{w}) = \overline{B(\bar{w}, w)} \\ \mathcal{C}(w, \bar{w}) &\rightarrow \mathcal{C}'(w, \bar{w}) = \bar{\mathcal{C}}(w, \bar{w}) = \overline{\mathcal{C}(\bar{w}, w)} \end{aligned} \quad (5.25)$$

Therefore, we may restrict to considering just a single branch of the solutions, specified by,

$$\kappa^2 \geq 0 \qquad R \leq 1 \qquad \mathcal{G} \geq 0 \qquad (5.26)$$

the other branch being related by complex conjugation.

5.5 Regularity conditions

To describe holographic duals to $4+1$ dimensional CFTs, we are mainly interested in solutions where the AdS_6 factor governs the entire non-compact part of the geometry. Therefore, we will assume that Σ is compact, with or without boundary. It will be convenient to examine the regularity conditions required in each one of these two cases separately.

When the Riemann surface Σ is compact and without boundary, a regular supergravity solution requires the metric functions f_2^2, f_6^2 and ρ^2 to remain strictly positive, and the axion-dilaton field to satisfy the strict inequality $|B| < 1$ throughout Σ . As a result, the corresponding conditions on the holomorphic data are given by the strict inequalities,

$$\kappa^2 > 0 \qquad R < 1 \qquad \mathcal{G} > 0 \qquad (5.27)$$

throughout Σ . The inequality $\kappa^2 > 0$ implies the strict inequality $|\lambda| < 1$.

When the Riemann surface Σ has a non-empty boundary $\partial\Sigma$, a regular supergravity solution may be obtained when the function f_2^2 vanishes on $\partial\Sigma$, provided such behavior corresponds to the shrinking of a sphere S^2 as part of a regular three-dimensional sub-manifold. It is clear from the explicit solutions in (5.8) that $9f_2^2 < f_6^2$ and therefore the vanishing of f_6^2 will force f_2^2 to vanish as well producing a space-time geometry with a short-distance singularity. To avoid such physically unacceptable singularities, we will assume henceforth that the regular part of the boundary of Σ is characterized by $f_2^2 = 0$ and $f_6^2 > 0$. In particular, a topologically non-trivial three-cycle, or three-sphere, may arise as part of the space-time manifold of a regular solution from fibering the sphere S^2 over a line segment on Σ which is spanned between two points on $\partial\Sigma$, and which cannot be continuously contracted to a point.

We will now investigate the behavior of a regular supergravity solution, and its associated holomorphic data, near a boundary point or segment of Σ . To analyze this behavior, we express f_2 and f_6 as directly as possible in terms of holomorphic data, and we find,

$$\frac{f_2^2}{f_6^2} = \frac{(1-R)^2}{9(1+R)^2} \qquad \frac{f_6^2}{c^2} = |\partial_w \mathcal{G}| \left(\frac{1-R}{\kappa^2} \right)^{\frac{1}{2}} \left(1 + \frac{1}{R} \right)^{\frac{1}{2}} \qquad (5.28)$$

The neighborhood of a regular boundary point is realized by letting $f_2^2 \rightarrow 0$ while keeping f_6^2 finite. On the variables R and κ^2 , this requires the following limiting behavior,

$$1 - R \rightarrow 0 \qquad \frac{1-R}{\kappa^2} \rightarrow \text{finite} \qquad (5.29)$$

The first condition ensures that $f_2/f_6 \rightarrow 0$, the second that f_6 stays finite. As a result, we have $\kappa^2 \rightarrow 0$ and $R \rightarrow 1$ at a regular boundary point, while their ratio stays finite.

Furthermore, the limiting behavior $R \rightarrow 1$ implies that $W \rightarrow 2 + \mathcal{O}((R-1)^2)$, which imposes a condition on the behavior of \mathcal{G} in view of the equation (5.7) for R . To derive this condition, we recast (5.7) as follows,

$$\frac{1-R}{\kappa^2} = \frac{6R\mathcal{G}}{(1-R)|\partial_w \mathcal{G}|^2} \quad (5.30)$$

Near a regular boundary point, the left side remains finite. Assuming that $\partial_w \mathcal{G}$ also remains finite, we are led to the following limiting behavior for \mathcal{G} ,

$$\frac{\mathcal{G}}{1-R} \rightarrow \text{finite} \quad (5.31)$$

Finally, we examine the regularity condition on the axion-dilaton field, namely $|B| < 1$ or equivalently $f^2 \geq 1$ and f^2 remains finite near a regular boundary point of $\partial\Sigma$. From equation (5.21) these conditions will be realized provided the following ratio has a finite limit,

$$\frac{Z^2 - \lambda}{1-R} \rightarrow \text{finite} \quad (5.32)$$

which gives us information on the behavior of the relative phase between λ and Z^2 . To analyze this condition, we make use of the definition $Z^2 = Re^{i\psi}$ and the relation $e^{i\psi} = \bar{\xi}/\xi = \bar{\mathcal{L}}/\mathcal{L}$, with \mathcal{L} defined in (4.49). Expressing the result further with the help of \mathcal{A}_\pm we obtain,

$$\frac{Z^2 - \lambda}{1-R} = \frac{\bar{\mathcal{A}}_- - \mathcal{A}_+}{\mathcal{L}} \times \frac{1 - |\lambda|^2}{1-R} \rightarrow \text{finite} \quad (5.33)$$

This condition is automatically fulfilled as long as $(\bar{\mathcal{A}}_- - \mathcal{A}_+)/\mathcal{L}$ remains finite.

In summary, the behavior of the supergravity fields near a boundary point on $\partial\Sigma$ is regular if and only if $\kappa^2 \rightarrow 0$ and the following two ratios have a finite and non-zero limit,

$$\frac{1-R}{\kappa^2} \rightarrow \text{finite} \quad \frac{\mathcal{G}}{\kappa^2} \rightarrow \text{finite} \quad (5.34)$$

When these conditions are obeyed, all supergravity fields are regular in the neighborhood of the corresponding regular boundary point. Of course, one may wish to consider supergravity solutions with sufficiently mild singularities, such as a diverging dilaton field at isolated points on the boundary. In this case, the condition \mathcal{G}/κ^2 being finite may have to be relaxed.

5.6 Recovering the T-dual of D4/D8

In this subsection, we will recover the T-dual of the D4/D8 solution given in [36] and show that this solution, even though singular, solves the BPS equations derived in the present paper. To find the T-dual of D4/D8, we consider a local coordinate system w, \bar{w} on a Riemann surface Σ , and make the following Ansatz for the holomorphic data,

$$\mathcal{A}_\pm = \frac{a}{2}w^2 \mp bw \quad (5.35)$$

with $a, b \in \mathbb{R}$. As a result, we find the following auxiliary quantities,

$$\kappa_\pm = aw \mp b \quad \kappa^2 = 2ab(w + \bar{w}) \quad \mathcal{B} = \frac{ab}{6}(1 - 2w^3) \quad (5.36)$$

The combinations \mathcal{G} and R , which are also required to construct the supergravity fields of the solutions, are given as follows,

$$\mathcal{G} = \frac{ab}{3Y^2} \quad R = \frac{Y-1}{Y+1} \quad Y = \frac{1}{\sqrt{1-(w+\bar{w})^3}} \quad (5.37)$$

The positivity conditions of (5.26) require $\kappa^2 \geq 0$, $\mathcal{G} \geq 0$ and $R \leq 1$, and thus,

$$0 < ab \quad 0 \leq w + \bar{w} < 1 \quad (5.38)$$

Note that neither $w + \bar{w} = 0$, nor $w + \bar{w} = 1$ satisfy the regularity conditions of (5.34), so that we must expect the resulting supergravity solution to have singularities there.

The metric functions are found as follows,

$$\rho^2 = \sqrt{2ab}(w+\bar{w})Y^{3/2} \quad f_6^2 = c_6^2 \sqrt{2ab}Y^{-1/2} \quad f_2^2 = \frac{c_6^2}{9} \sqrt{2ab}Y^{-5/2} \quad (5.39)$$

For the axion and dilaton we use the formulas in appendix E, eqs. (E.11) and (E.12), to get them separately right away. This yields,

$$e^{-2\phi} = \frac{2bY}{a(w+\bar{w})} \quad \chi = \frac{ia}{2b}(w-\bar{w}) \quad (5.40)$$

To match to [36], we change from w, \bar{w} to real coordinates θ, ϕ_3 defined by,

$$\cos \theta = (w + \bar{w})^{3/2} \quad \phi_3 = \frac{ia}{2bm}(w - \bar{w}) \quad (5.41)$$

and fix the parameters a, b, c_6 by the following choice,

$$a = \frac{27}{16}m^{1/3} \quad b = \frac{9}{8m^{1/3}} \quad c_6 = 1 \quad (5.42)$$

With the notation $\widetilde{W} = (m \cos \theta)^{-1/6}$, the axion and dilaton fields become,

$$e^{-2\phi} = \frac{3 \sin \theta}{4\widetilde{W}^4} \quad \chi = m\phi_3 \quad (5.43)$$

This reproduces the results of (A.1) of [36], noting that the definition of the dilaton in (2.7) differs from that used in [36] by a factor 2. The metric functions become,

$$4f_6^2 e^{-\phi} = \widetilde{W}^2 \quad 4f_2^2 e^{-\phi} = \widetilde{W}^2 \sin^2 \theta \quad 4\rho^2 e^{-\phi} |dw|^2 = \widetilde{W}^2 \left(d\theta^2 + \frac{4d\phi_3^2}{\widetilde{W}^4 \sin^2 \theta} \right) \quad (5.44)$$

Our metric is in Einstein frame, that of [36] in string frame. The metric functions in string frame are simply given by dropping the dilaton factors in the above expressions, i.e. $(f_6^2)_{\text{string}} = e^\phi f_6^2$ etc. Upon this change to string frame, we have exactly the metric in (A.1) of [36]. This just leaves the flux field $C_{(2)}$, which we obtain from (5.10) as,

$$C_{(2)} = \frac{ic_6^2}{9} \left(6aw\bar{w} - a(w+\bar{w})^2(3+Y^{-2}) + 6b(w-\bar{w}) - \mathcal{K}_0 \right) \hat{e}^{67} \quad (5.45)$$

Switching coordinates as in (5.41), we find,

$$dC_{(2)} = \frac{5i}{8}(m \cos \theta)^{1/3} \sin^3 \theta d\theta \wedge \hat{e}^{67} + (1 + im\phi_3)d\phi_3 \wedge \hat{e}^{67} \quad (5.46)$$

To compare to [36], we separate out the real and imaginary parts, which are,

$$\begin{aligned} \text{Re}(dC_{(2)}) &= dB_{(2)} = d\phi_3 \wedge \hat{e}^{67} \\ \text{Im}(dC_{(2)}) &= \frac{5}{8}(m \cos \theta)^{1/3} \sin^3 \theta d\theta \wedge \hat{e}^{67} + m\phi_3 d\phi_3 \wedge \hat{e}^{67} \end{aligned} \quad (5.47)$$

The combination which appears as F_3 in (A.1) of [36] becomes,

$$\text{Im}(dC_{(2)}) - \chi dB_{(2)} = \frac{5}{8}(m \cos \theta)^{1/3} \sin^3 \theta d\theta \quad (5.48)$$

So with the choice (5.35), (5.42) for the holomorphic data we reproduce the T-dual of the D4/D8 solution exactly. As detailed in the introduction, this solution is singular. But the fact that we recover it from our general local solution provides a useful consistency check.

5.7 Satisfying positivity and regularity conditions locally near $\partial\Sigma$

In this subsection, we shall show that, at least locally in a finite neighborhood of part of the boundary of Σ , it is possible to satisfy both positivity and regularity conditions. By a conformal transformation, we map a boundary component of Σ to the real axis, and take the interior of Σ to be part of the upper half plane parametrized by complex coordinates w, \bar{w} . To realize the positivity condition $\kappa^2 = 0$, or equivalently $|\lambda| = 1$, on $\partial\Sigma$ we choose $\lambda = (1 + iw)/(1 - iw)$, so that the functions \mathcal{A}_\pm are related by the following equation,

$$\partial_w \mathcal{A}_+(w) = \frac{1 + iw}{1 - iw} \partial_w \mathcal{A}_-(w) \quad (5.49)$$

We restrict attention to local solutions for which $\partial_w \mathcal{A}_\pm$ are rational functions of w , and can be decomposed into a sum of simple poles plus a constant additive term,

$$\partial_w \mathcal{A}_\pm(w) = (1 \pm iw) \sum_{n=1}^N \frac{a_n}{w - x_n} \quad (5.50)$$

The positions of the poles x_n are chosen to be real. We guarantee the absence of zeros for $\partial_w \mathcal{A}_-$ in the upper half plane by requiring $a_n e^{-i\phi} \in \mathbb{R}$ for all n and some n -independent phase ϕ . The integrals \mathcal{A}_\pm are as follows,

$$\mathcal{A}_\pm(w) = \alpha_\pm + \sum_{n=1}^N a_n \left((1 \pm ix_n) \ln(w - x_n) \pm iw \right) \quad (5.51)$$

where α_\pm are complex constants. For $N \geq 2$, the calculation of \mathcal{B} reveals the presence of dilogarithms. To avoid this complication, we shall examine only the simplest case $N = 1$ where no dilogarithms appear in \mathcal{B} . Setting $a_1 = a$ and $x_1 = x$, we then obtain,

$$\kappa^2 = \frac{-2i(w - \bar{w})|a|^2}{|w - x|^2} \quad (5.52)$$

which indeed vanishes on \mathbb{R} and is strictly positive in the upper half plane. Integrating these equations as well as the one for \mathcal{B} , we find,

$$\begin{aligned}\mathcal{A}_{\pm}(w) &= \alpha_{\pm} \pm ia(w-x) + a(1 \pm ix) \ln(w-x) \\ \mathcal{B}(w) &= \mathcal{B}_0 + ia(w-x)(4a - \alpha_+ - \alpha_-) \\ &\quad + a\left(\alpha_+(1-ix) - \alpha_-(1+ix) - 2ia(w-x)\right) \ln(w-x)\end{aligned}\quad (5.53)$$

where \mathcal{B}_0 is an integration constant. We use these ingredients to compute \mathcal{G} . The vanishing of \mathcal{G} on the real axis requires $\mathcal{B}_0 + \bar{\mathcal{B}}_0 = 0$ as well as $a^2 = -|a|^2$ and $\alpha_- = \bar{\alpha}_+$, and we obtain,

$$\mathcal{G}(w) = -2i|a|^2(w - \bar{w})\left(2 - \ln|w-x|^2\right) \quad (5.54)$$

In the upper half plane the pre-factor $-2i(w - \bar{w})$ is positive, so that we have $\mathcal{G} > 0$ inside a semicircle centered at x with $\text{Im}(w) > 0$ and $\ln|w-x|^2 < 2$, while $\mathcal{G} = 0$ on the real axis.

Therefore, we have established that it is possible, locally in a finite region near a boundary component, to satisfy both the positivity and regularity conditions on κ^2 and \mathcal{G} . On the semi-circle defined by $\ln|w-x|^2 = 2$ the behavior fails to be regular, since we have $\mathcal{G} = 0$ but $\kappa^2 > 0$ reminiscent of the singularity of the T-dual to D4/D8 at $w + \bar{w} = 1$. It remains to find supergravity solutions which obey the positivity and regularity conditions globally, an investigation that we shall reserve for future work.

5.8 Supergravity solutions with monodromy

When Σ is a compact Riemann surface without boundary, and the locally holomorphic functions \mathcal{A}_{\pm} are assumed to be single-valued on Σ , then they must be constant, which does not produce any supergravity solutions. Therefore, on any compact Σ without boundary, the existence of regular supergravity solutions will require the locally holomorphic functions \mathcal{A}_{\pm} to have non-trivial monodromy, or poles which conspire in such a way that the supergravity fields remain finite.

Under the weaker assumption that the locally holomorphic function λ is single-valued on a compact Riemann surface Σ without boundary, but \mathcal{A}_{\pm} are allowed a non-trivial monodromy, it follows from the condition $|\lambda| < 1$ that λ must be constant. The relation between κ_{\pm} may then be integrated explicitly, and we have,

$$\mathcal{A}_+ = \lambda \mathcal{A}_- + a_0 \quad (5.55)$$

where a_0 is an arbitrary complex constant. The monodromies of (5.12), which we repeat here for convenience,

$$\begin{aligned}\mathcal{A}_+ &\rightarrow \mathcal{A}'_+ = +u\mathcal{A}_+ - v\mathcal{A}_- + a_+ \\ \mathcal{A}_- &\rightarrow \mathcal{A}'_- = -\bar{v}\mathcal{A}_+ + \bar{u}\mathcal{A}_- + a_-\end{aligned}\quad (5.56)$$

will be compatible with the relation $\mathcal{A}_+ = \lambda \mathcal{A}_- + a_0$ provided $\mathcal{A}'_+ = \lambda \mathcal{A}'_- + a_0$ as well. This condition in turn imposes two complex-valued conditions on the three complex-valued monodromy parameters $u, v, a_+ = \bar{a}_-$, given as follows,

$$\begin{aligned}u\lambda - \bar{u}\lambda + v + \bar{v}\lambda^2 &= 0 \\ (1 - u - \lambda\bar{v})a_0 - a_+ + \lambda\bar{a}_+ &= 0\end{aligned}\quad (5.57)$$

Manifestly, the only solution to these equations with trivial $SU(1,1)$ monodromy, namely $u = 1, v = 0$ are solutions with no monodromy at all since $a_+ - \lambda \bar{a}_+ = 0$ implies $a_+ = 0$ in view of $|\lambda| < 1$. Thus, solutions with constant λ must necessarily involve a non-trivial $SU(1,1)$ monodromy on \mathcal{A}_\pm . The existence of such solutions will be investigated in detail in a subsequent publication.

Allowing all three locally holomorphic functions \mathcal{A}_\pm and λ to have non-trivial monodromy under $SU(1,1) \times \mathbb{C}$, the mathematical problem becomes quite interesting, and quite involved. The pair $(\mathcal{A}_+, \mathcal{A}_-)$ may then be viewed as a holomorphic section of a holomorphic bundle over Σ with a structure group which is a subgroup of $SU(1,1) \times \mathbb{C}$. The fibers of this bundle are subject to the regularity conditions $\kappa^2 > 0$ and $\mathcal{G} > 0$.

The simplest situation is when Σ is compact and without boundary, and we shall attempt to give a plausible mathematical context for this case. The $SU(1,1) \times \mathbb{C}$ invariance of \mathcal{G} then guarantees the continuity of \mathcal{G} as a function on Σ and therefore its boundedness. Using the freedom to shift \mathcal{B} by an arbitrary constant, we may always adjust the shift to make $\mathcal{G} > 0$, and therefore to render this condition trivially satisfied. The only remaining condition on the holomorphic sections $(\mathcal{A}_+, \mathcal{A}_-)$ of the bundle is then a condition on its associated one-forms,

$$|\partial_w \mathcal{A}_+|^2 - |\partial_w \mathcal{A}_-|^2 < 0 \quad (5.58)$$

The construction of this bundle should be expected to parallel the construction of vector bundles over Σ with structure group contained in $SU(1,1)$, with the important difference that in our case the space of one-forms $\partial_w \mathcal{A}_\pm$ is subject to the above inequality which makes each fiber into a solid cone in \mathbb{C}^2 rather than a vector space.

Holomorphic vector bundles over compact Riemann surfaces were considered early on in [37] and classified when their structure group is a subgroup of $SU(n)$ and the bundle satisfies certain stability conditions [38, 39].⁷ The equivalence classes of rank n holomorphic vector bundles are then in one-to-one correspondence with the irreducible representations of the fundamental homotopy group $\pi_1(\Sigma)$ into $SU(n)$. The concrete realization of these representations on a compact surface of genus $g \geq 2$ is constructed as follows. We begin by introducing a basis of A_i and B_i cycles for the first homology group $H_1(\Sigma, \mathbb{Z})$, with canonical normalization of their intersection matrix $\#(A_i, A_j) = \#(B_i, B_j) = 0$ and $\#(A_i, B_j) = \delta_{ij}$ for $i, j = 1, \dots, g$. The representation γ of $\pi_1(\Sigma)$ into $SU(n)$ may then be concretely described by assigning elements $\gamma(A_i)$ and $\gamma(B_i)$ in $SU(n)$ to the homology cycles A_i and B_i respectively, subject to the standard closure condition on the commutators,

$$\prod_{i=1}^g \gamma(A_i) \gamma(B_i) \gamma(A_i)^{-1} \gamma(B_i)^{-1} = I \quad (5.59)$$

The product in (5.59) is ordered in the index i , and I stands for the identity matrix in $SU(n)$. The discrete group Γ is then freely generated by the elements $\gamma(A_i)$ and $\gamma(B_i)$ for $i = 1, \dots, g$, subject to the closure relation (5.59).

⁷A useful introduction to this work, accessible to physicists, may be found in [40]. We thank David Gieseker for pointing us to his paper.

Holomorphic vector bundles over Σ whose structure group Γ is a subgroup of $\mathrm{SL}(2, \mathbb{R}) = \mathrm{SU}(1, 1)$ instead should admit an analogous construction. We should assign to each homology generator A_i, B_i a transformation $\gamma(A_i), \gamma(B_i)$ in Γ subject to the closure relations of (5.59). The corresponding group Γ is then freely generated by these elements, just as was the case for the unitary groups. In the simplest case where the structure group is \mathbb{Z}_2 , the corresponding differential forms span the space of Prym differentials [41] on the surface Σ , familiar from \mathbb{Z}_2 orbifold constructions in string theory. The case of a more general structure group Γ may be viewed as a generalization of Prym differentials to the case of un-ramified covers of Σ with higher structure group Γ . Finally, for our set-up the fibers are solid cones of one-forms in \mathbb{C}^2 rather than vector spaces, but the construction of such bundles is expected to follow in parallel. A detailed investigation into these possibilities will be relegated to future work.

6 Discussion

Supersymmetric solutions to Type IIB supergravity with an AdS_6 factor and the corresponding isometries are of considerable interest to describe holographically the still mysterious 4+1 dimensional CFTs discovered in [5, 6] and realized in string theory via (p, q) brane webs. As quantum field theories these CFTs are defined only indirectly, as the non-trivial UV fixed points of certain 4+1 dimensional super-Yang-Mills theories. The description as Yang-Mills theories, however, is non-renormalizable and only captures the IR effective action of a relevant deformation of the CFT. Having supergravity duals at our disposal would allow for extensive quantitative studies of the actual UV CFTs, which from the field theory side are hard to do even in principle without a known Lagrangian description.

In this work we have constructed the general local solution to Type IIB supergravity with $\mathrm{SO}(2, 5) \times \mathrm{SO}(3)$ isometry and 16 supersymmetries, which are precisely the symmetries expected for holographic duals to these 4+1 dimensional SCFTs. The local solutions are constructed from AdS_6 and S^2 spaces, which realize the desired isometries and are fibered over a Riemann surface Σ . With the supergravity fields spelled out in section 2, the local solutions are given in terms of two locally holomorphic functions \mathcal{A}_\pm , as summarized in section 5.1 and 5.2. The solutions transform properly under $\mathrm{SL}(2, \mathbb{R})$, and this symmetry serves as an organizing guide throughout the derivations. Finally, we recover the (singular) T-dual of the D4/D8 solution in Type IIA as a special case.

A crucial ingredient which has allowed us to go beyond earlier works, where the BPS equations were reduced to a set of coupled PDEs [16–18], was keeping the freedom to choose convenient coordinates on the Riemann surface Σ . Separating holomorphic and anti-holomorphic dependences in conformally flat coordinates on Σ featured prominently in the derivations and allowed us to actually solve the reduced BPS equations. The complete local solution provides the basic building blocks for constructing globally regular solutions with the aforementioned isometries, and is a significant step towards understanding and eventually classifying supergravity duals for the 4+1 dimensional CFTs realized by (p, q) brane webs.

The natural next question is whether or not there are globally regular solutions with the symmetries discussed above. While the existence of a large- N limit on the CFT side suggests that there should be dual supergravity solutions, the difficulties in finding such solutions so far call for a more careful perspective. Indeed, there are known examples where a superconformal field theory with a large- N limit does not admit a dual description which reduces to supergravity alone, even at large N and strong coupling [42]. The existence of a supergravity dual is obstructed in that case by the absence of a gap in the spectrum to isolate a small number of states with low scaling dimension. As a result, the dual description always involves stringy states. In the example of [42], however, realizing a superconformal theory at large N needs a similarly large number of flavors, of the same order as N . This Veneziano limit is different from the usual 't Hooft limit and from the case we are looking at here, where superconformal theories exist also with small numbers of flavors. There is, quite on the contrary, an upper limit on the number of flavor multiplets in the CFTs we are attempting to find a dual description for. The large- N limit should therefore indeed be a 't Hooft limit, and these arguments suggest that finding a dual supergravity description should be possible. We discussed regularity conditions in section 5, but constructing globally regular solutions is still a non-trivial task, and we plan to come back to it in the future.

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A Clifford algebra basis adapted to the Ansatz

The signature of the space-time metric is chosen to be $(- + \cdots +)$. The Dirac-Clifford algebra is defined by $\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}I_{32}$. We choose a basis for the Clifford algebra which is well-adapted to the $AdS_6 \times S^2 \times \Sigma$ Ansatz, with the frame labeled as in (2.14),

$$\begin{aligned}
 \Gamma^m &= \gamma^m \otimes I_2 \otimes I_2 & m &= 0, 1, 2, 3, 4, 5 \\
 \Gamma^i &= \gamma_{(1)} \otimes \gamma^i \otimes I_2 & i &= 6, 7 \\
 \Gamma^a &= \gamma_{(1)} \otimes \sigma^3 \otimes \gamma^a & a &= 8, 9
 \end{aligned} \tag{A.1}$$

where a convenient basis for the lower dimensional Dirac-Clifford algebras is as follows,

$$\begin{aligned}
 \gamma^0 &= -i\sigma^2 \otimes I_2 \otimes I_2 \\
 \gamma^1 &= \sigma^1 \otimes I_2 \otimes I_2 \\
 \gamma^2 &= \sigma^3 \otimes \sigma^2 \otimes I_2 \\
 \gamma^3 &= \sigma^3 \otimes \sigma^1 \otimes I_2 \\
 \gamma^4 &= \sigma^3 \otimes \sigma^3 \otimes \sigma^1 \\
 \gamma^5 &= \sigma^3 \otimes \sigma^3 \otimes \sigma^2 \\
 \gamma^6 &= \sigma^1 \\
 \gamma^7 &= \sigma^2 \\
 \gamma^8 &= \sigma^1 \\
 \gamma^9 &= \sigma^2
 \end{aligned} \tag{A.2}$$

We will also need the chirality matrices on the various components of $AdS_6 \times S^2 \times \Sigma$, and they are chosen as follows,

$$\begin{aligned}
 \gamma_{(1)} &= \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \\
 \gamma_{(2)} &= \sigma^3 \\
 \gamma_{(3)} &= \sigma^3
 \end{aligned} \tag{A.3}$$

so that,

$$\begin{aligned}
 \Gamma^{012345} &= -\gamma_{(1)} \otimes I_2 \otimes I_2 \\
 \Gamma^{67} &= i I_6 \otimes \gamma_{(2)} \otimes I_2 \\
 \Gamma^{89} &= i I_6 \otimes I_2 \otimes \gamma_{(3)}
 \end{aligned} \tag{A.4}$$

The 10-dimensional chirality matrix in this basis is given by,

$$\Gamma^{11} = \Gamma^{0123456789} = \gamma_{(1)} \otimes \gamma_{(2)} \otimes \gamma_{(3)} = \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \tag{A.5}$$

The complex conjugation matrices in each component are defined by,

$$\begin{aligned}
 (\gamma^m)^* &= +B_{(1)}\gamma^m B_{(1)}^{-1} & (B_{(1)})^* B_{(1)} &= -I_6 & B_{(1)} &= -i\gamma^2\gamma^5 = I_2 \otimes \sigma_1 \otimes \sigma_2 \\
 (\gamma^i)^* &= -B_{(2)}\gamma^i B_{(2)}^{-1} & (B_{(2)})^* B_{(2)} &= -I_2 & B_{(2)} &= \gamma^7 = \sigma^2 \\
 (\gamma^a)^* &= -B_{(3)}\gamma^a B_{(3)}^{-1} & (B_{(3)})^* B_{(3)} &= -I_2 & B_{(3)} &= \gamma^9 = \sigma^2
 \end{aligned} \tag{A.6}$$

where in the last column we have also listed the form of these matrices in our particular basis. The 10-dimensional complex conjugation matrix \mathcal{B} satisfies,

$$(\Gamma^M)^* = \mathcal{B}\Gamma^M\mathcal{B}^{-1} \qquad \mathcal{B}^*\mathcal{B} = I_{32} \qquad [\mathcal{B}, \Gamma^{11}] = 0 \tag{A.7}$$

and in this basis is given by,

$$\begin{aligned}
 \mathcal{B} &= -iB_{(1)} \otimes (B_{(2)}\gamma_{(2)}) \otimes B_{(3)} \\
 &= I_2 \otimes \sigma^1 \otimes \sigma^2 \otimes \sigma^1 \otimes \sigma^2
 \end{aligned} \tag{A.8}$$

B The geometry of Killing spinors

In this appendix, we review the relation between the Killing spinor equation and the parallel transport equation in the presence of a flat connection with torsion on S^2 and AdS_4 .

B.1 The sphere S^2

On the sphere S^2 with unit radius, the Killing spinor equation is given by,

$$\left(\nabla_i + \eta \frac{1}{2} \sigma^i \sigma^3\right) \varepsilon = 0 \quad i = 1, 2 \quad (\text{B.1})$$

where $\sigma^1, \sigma^2, \sigma^3$ are the standard Pauli matrices, and ∇_i is the spin connection on the sphere S^2 with unit radius, and the derivative is expressed with respect to frame indices i . An equivalent equation is obtained by letting $\sigma^i \sigma^3 \rightarrow i \sigma^i$ and $\varepsilon' = e^{-i\pi\sigma^3/4} \varepsilon$. Integrability of this system of equations requires $\eta = \pm 1$. The system may be solved by restricting the canonical flat connection on $\text{SU}(2)$, as was reviewed in an appendix of [19]. Here, we will take a more direct approach to obtaining an explicit solution.

We parametrize the sphere S^2 with unit radius by angles θ_1, θ_2 and take the following conventions for the round metric,

$$ds^2 = d\theta_2^2 + \sin^2 \theta_2 d\theta_1^2 \quad (\text{B.2})$$

The frame and the spin connection are given by,

$$e^1 = \sin \theta_2 d\theta_1 \quad e^2 = d\theta_2 \quad \omega^{12} = \cos \theta_2 d\theta_1 \quad (\text{B.3})$$

and satisfy the vanishing torsion condition,

$$de^a + \omega^a_b \wedge e^b = 0 \quad (\text{B.4})$$

The Killing spinor equation on the sphere S^2 of unit radius is of the form,

$$\left(\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab}\right) \varepsilon = \eta \frac{i}{2} e_\mu^a \gamma_a \varepsilon \quad (\text{B.5})$$

with $\eta = \pm 1$ required by integrability. With the Dirac matrices defined in appendix A we have $\gamma^1 = \sigma_1$ and $\gamma^2 = \sigma_2$, and this equation becomes,

$$\partial_{\theta_1} \varepsilon + \frac{i}{2} \cos \theta_2 \sigma_3 \varepsilon = \frac{i}{2} \eta \sin \theta_2 \sigma_1 \varepsilon \quad \partial_{\theta_2} \varepsilon = \frac{i}{2} \eta \sigma_2 \varepsilon \quad (\text{B.6})$$

and is solved by,

$$\varepsilon(\eta) = \exp\left(\frac{i}{2} \eta \theta_2 \sigma_2\right) \exp\left(-\frac{i}{2} \theta_1 \sigma_3\right) \varepsilon_0 \quad (\text{B.7})$$

Here, ε_0 is an arbitrary constant spinor. Hence the space of Killing spinors is two-dimensional. We can verify the following statements for a Killing spinor $\varepsilon(\eta)$ which satisfies the Killing equation with a given $\eta = \pm 1$,

1. $\gamma_{(2)} \varepsilon(\eta) = \sigma_3 \varepsilon(\eta)$ satisfies the Killing equation with $\eta \rightarrow -\eta$;
2. $\sigma_1 \varepsilon(\eta)^*$ satisfies the Killing equation with $\eta \rightarrow -\eta$;
3. $\sigma_2 \varepsilon(\eta)^*$ satisfies the Killing equation with $\eta \rightarrow +\eta$.

B.2 Minkowski AdS_6

The construction based on canonical connections given in [19] for S^2 may be generalized to all spheres and their hyperbolic AdS counterparts. Here, we present the case of Minkowski signature $AdS_6 = SO(2, 5)/SO(1, 5)$. The Dirac-Clifford algebra of $SO(2, 5)$ is built from the Clifford generators γ^μ , of the Lorentz group $SO(1, 5)$,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad \eta = \text{diag}[- + + + + +] \quad (\text{B.8})$$

for $\mu, \nu = 0, 1, 2, 3$, supplemented with the chirality matrix, $\gamma^\sharp = i\gamma_{(1)}$ (the factor of i being required since the \sharp dimension has negative entry in the metric),

$$\{\gamma^{\bar{\mu}}, \gamma^{\bar{\nu}}\} = 2\bar{\eta}^{\bar{\mu}\bar{\nu}} \quad \bar{\eta} = \text{diag}[- - + + + +] \quad (\text{B.9})$$

for $\bar{\mu}, \bar{\nu} = \sharp, 0, 1, 2, 3, 4, 5$. The corresponding Maurer-Cartan form, expressed in the spinor representation, on $SO(2, 5)$ is given by,

$$\omega^{(t)} = V^{-1}dV = \frac{1}{4}\omega_{\bar{\mu}\bar{\nu}}^{(t)}\gamma^{\bar{\mu}\bar{\nu}} \quad V \in \text{Spin}(2, 5) \quad (\text{B.10})$$

It obviously satisfies the Maurer-Cartan equations, $d\omega^{(t)} + \omega^{(t)} \wedge \omega^{(t)} = 0$. We decompose $\omega^{(t)}$ onto the $SO(1, 5)$ and AdS_6 directions of cotangent space,

$$\omega^{(t)} = \frac{1}{4}\omega_{\mu\nu}\gamma^{\mu\nu} + \frac{i}{2}e_\mu\gamma^\mu\gamma_{(1)} \quad \begin{cases} \omega_{\mu\nu} \equiv \omega_{\mu\nu}^{(t)} & \mu, \nu = 0, 1, 2, 3, 4, 5 \\ e_\mu \equiv \omega_{\mu\sharp}^{(t)} & \mu = 0, 1, 2, 3, 4, 5 \end{cases} \quad (\text{B.11})$$

The Maurer-Cartan equations $d\omega^{(t)} + \omega^{(t)} \wedge \omega^{(t)} = 0$ for $\omega^{(t)}$ imply the absence of torsion and the constancy of curvature. The Killing spinor equation coincides with the equation for parallel transport,

$$(d + V^{-1}dV)\varepsilon = \left(d + \frac{1}{4}\omega_{\mu\nu}\gamma^{\mu\nu} + \frac{i}{2}\eta e_\mu\gamma^\mu\gamma_{(1)}\right)\varepsilon = 0 \quad (\text{B.12})$$

For $\eta = +1$, the general solution is given by $\varepsilon_+ = V^{-1}\varepsilon_0$ and ε_0 is constant, while for $\eta = -1$, the solution is $\varepsilon_- = \gamma_{(1)}\varepsilon_+$. An equivalent form of the Killing spinor equation, in a basis in which the chirality matrix $\gamma_{(1)}$ is eliminated, is achieved by making the transformation,

$$\varepsilon = \exp\left(\frac{i\pi}{4}\gamma_{(1)}\right)\varepsilon' \quad (\text{B.13})$$

In terms of ε' , we recover the first equation in (3.1).

B.2.1 Explicit form of the Killing spinors

We parametrize AdS_6 with unit radius by a radial coordinate r and $x^i \in \mathbb{R}$ for $i = 1, 2, 3, 4$, and use the $SO(2, 5)$ -invariant metric,

$$ds^2 = dr^2 + e^{2r} \left(-dt^2 + \sum_{i=1}^4 dx_i^2\right) \quad (\text{B.14})$$

The Killing spinor equation is given by,

$$D_\mu \varepsilon = \frac{\eta}{2} \gamma_\mu \varepsilon \quad (\text{B.15})$$

with $\eta = \pm 1$. The frame and spin connection are given by,

$$e^r = dr \quad e^i = e^r dx^i \quad \omega^{ir} = e^r dx^i \quad (\text{B.16})$$

Hence the Killing spinor equation becomes,

$$\partial_r \varepsilon = \frac{\eta}{2} \gamma_r \varepsilon \quad \partial_i \varepsilon + \frac{1}{2} e^r \gamma_{ir} \varepsilon = \frac{\eta}{2} e^r \gamma_i \varepsilon \quad (\text{B.17})$$

The general solution for the Killing spinor equation is,

$$\varepsilon = e^{\frac{\eta}{2} r \gamma_r} \left(1 + \frac{1}{2} x^i \gamma_i (\eta - \gamma_r) \right) \varepsilon_0 \quad (\text{B.18})$$

where ε_0 is a constant spinor, hence the space of Killing spinors of AdS_6 is eight dimensional. We can label the basis vectors by $\eta = \pm 1$ which each have a four dimensional degeneracy which will play no role in the following other than leading to the correct number of preserved supersymmetries in the end. We can verify the following statements for an AdS_6 Killing spinor $\epsilon(\eta)$ which satisfies the Killing equation with a given $\eta = \pm 1$,

1. $\gamma_{(1)} \epsilon(\eta) = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \epsilon(\eta)$ satisfies the Killing equation with $\eta \rightarrow -\eta$;
2. $B_{(1)} \epsilon(\eta)^* = 1_2 \otimes \sigma_2 \otimes \sigma_1 \epsilon(\eta)^*$ satisfies the Killing equation with $\eta \rightarrow \eta$;
3. $\tilde{B}_{(1)} \epsilon(\eta)^* = \sigma_3 \otimes \sigma_1 \otimes \sigma_2 \epsilon(\eta)^*$ satisfies the Killing equation with $\eta \rightarrow -\eta$.

Note that we choose the complex conjugation matrix to be $B_{(1)}$ in appendix [A](#)

C Derivation of the BPS equations

We begin by collecting some identities that will be useful during the reduction of the BPS equations. We will also need the following decompositions of ε and $\mathcal{B}^{-1} \varepsilon^*$,

$$\varepsilon = \sum_{\eta_1, \eta_2, \eta_3} \chi^{\eta_1, \eta_2} \otimes \zeta_{\eta_1, \eta_2} \quad \mathcal{B}^{-1} \varepsilon^* = \sum_{\eta_1, \eta_2} \chi^{\eta_1, \eta_2} \otimes \star \zeta_{\eta_1, \eta_2} \quad (\text{C.1})$$

where we use the abbreviations,

$$\star \zeta_{\eta_1, \eta_2} = -i \sigma^2 \eta_2 \zeta_{\eta_1, -\eta_2}^* \quad \star \zeta = \tau^{02} \otimes \sigma^2 \zeta^* \quad (\text{C.2})$$

in τ -matrix notation. In addition, we have the chirality relations,

$$\sigma^3 \zeta_{\eta_1, \eta_2} = -\zeta_{-\eta_1, -\eta_2} \quad \tau^{11} \otimes \sigma^3 \zeta = -\zeta \quad (\text{C.3})$$

C.1 The dilatino equation

The dilatino equation is,

$$0 = iP_A \Gamma^A \mathcal{B}^{-1} \varepsilon^* - \frac{i}{24} \Gamma \cdot G \varepsilon \quad (\text{C.4})$$

Reduced to the Ansatz of subsection 2.3, we have the following simplifications,

$$\begin{aligned} P_A \Gamma^A &= p_a \Gamma^a \\ \Gamma \cdot G &= 3! g_a \Gamma^{67a} = 6g_a \Gamma^a i I_8 \otimes \gamma_{(2)} \otimes I_2 \end{aligned} \quad (\text{C.5})$$

The dilatino equation now becomes,

$$0 = ip_a \Gamma^a \sum_{\eta_1, \eta_2} \chi^{\eta_1, \eta_2} \otimes \star \zeta_{\eta_1, \eta_2} + \frac{1}{4} g_a \Gamma^a \sum_{\eta_1, \eta_2} \chi^{\eta_1, -\eta_2} \otimes \zeta_{\eta_1, \eta_2} \quad (\text{C.6})$$

from which we extract the equation satisfied by the ζ -spinors,

$$0 = ip_a \gamma^a \star \zeta_{\eta_1, \eta_2} + \frac{1}{4} g_a \gamma^a \zeta_{\eta_1, -\eta_2} \quad (\text{C.7})$$

Using the explicit expression for $\star \zeta$ and reversing the sign of η_2 , we find,

$$0 = -p_a \gamma^a \eta_2 \sigma^2 \zeta_{\eta_1, \eta_2}^* + \frac{1}{4} g_a \gamma^a \zeta_{\eta_1, \eta_2} \quad (\text{C.8})$$

Recasting this equation in terms of the τ -notations, we have,

$$0 = -4p_a \tau^{(03)} \gamma^a \sigma^2 \zeta^* + g_a \gamma^a \zeta \quad (\text{C.9})$$

C.2 The gravitino equation

The gravitino equation is,

$$\begin{aligned} 0 &= d\varepsilon + \omega \varepsilon - \frac{i}{2} Q \varepsilon + \mathfrak{g} \mathcal{B}^{-1} \varepsilon^* \\ \omega &= \frac{1}{4} \omega_{AB} \Gamma^{AB} \\ \mathfrak{g} &= -\frac{1}{96} e_A \left(\Gamma^A (\Gamma \cdot G) + 2(\Gamma \cdot G) \Gamma^A \right) \end{aligned} \quad (\text{C.10})$$

where A, B are the 10-dimensional frame indices.

C.2.1 The calculation of $(d + \omega)\varepsilon$

The spin connection components are ω^a_b , whose explicit form we will not need and,

$$\begin{aligned} \omega^m_n &= \hat{\omega}^m_n & \omega^m_a &= e^m \frac{\partial_a f_6}{f_6} \\ \omega^i_j &= \hat{\omega}^i_j & \omega^i_a &= e^i \frac{\partial_a f_2}{f_2} \end{aligned} \quad (\text{C.11})$$

The hats refer to the canonical connections on AdS_4 and S^2 respectively. Projecting the spin-connection along the various directions we have,

$$\begin{aligned}
 (m) \quad & \nabla'_m \varepsilon + \frac{D_a f_6}{2f_6} \Gamma_m \Gamma^a \varepsilon \\
 (i) \quad & \nabla'_i \varepsilon + \frac{D_a f_2}{2f_2} \Gamma_i \Gamma^a \varepsilon \\
 (a) \quad & \nabla_a \varepsilon
 \end{aligned} \tag{C.12}$$

where the prime on the covariant derivative indicates that only the connection along AdS_4 and S^2 respectively is included. Using the Killing spinor equations (3.1) we can eliminate the primed covariant derivatives, which yields,

$$\begin{aligned}
 (m) \quad & \frac{1}{2f_6} \Gamma_m \sum_{\eta_1, \eta_2} \eta_1 \chi^{\eta_1, \eta_2} \otimes \zeta_{\eta_1, \eta_2} + \frac{D_a f_6}{2f_6} \Gamma_m \Gamma^a \varepsilon \\
 (i) \quad & \frac{i}{2f_2} \Gamma_i \sum_{\eta_1, \eta_2} \eta_2 \chi^{-\eta_1, \eta_2} \otimes \zeta_{\eta_1, \eta_2} + \frac{D_a f_2}{2f_2} \Gamma_i \Gamma^a \varepsilon
 \end{aligned} \tag{C.13}$$

Using the equation $\Gamma^a = \gamma_{(1)} \otimes \gamma_{(2)} \otimes \gamma_{(3)} \otimes \gamma^a$, we have,

$$\begin{aligned}
 (m) \quad & \Gamma_m \sum_{\eta_1, \eta_2} \chi^{\eta_1, \eta_2} \otimes \left(\frac{1}{2f_6} \eta_1 \zeta_{\eta_1, \eta_2} + \frac{D_a f_6}{2f_6} \gamma^a \zeta_{-\eta_1, -\eta_2} \right) \\
 (i) \quad & \Gamma_i \sum_{\eta_1, \eta_2} \chi^{\eta_1, \eta_2} \otimes \left(\frac{i}{2f_2} \eta_2 \zeta_{-\eta_1, \eta_2} + \frac{D_a f_2}{2f_2} \gamma^a \zeta_{-\eta_1, -\eta_2} \right)
 \end{aligned} \tag{C.14}$$

where we have pulled a factor of Γ_M out front. It will turn out that all terms in the gravitino equation contain $\Gamma_M \chi^{\eta_1, \eta_2, \eta_3}$, and we will require the coefficients to vanish independently, just as we did for the dilatino equation. The coefficient of $\Gamma_M \chi^{\eta_1, \eta_2, \eta_3}$ can be expressed in the τ -matrix notation as,

$$\begin{aligned}
 (m) \quad & \frac{1}{2f_6} \tau^{(30)} \zeta + \frac{D_a f_6}{2f_6} \tau^{(11)} \gamma^a \zeta \\
 (i) \quad & \frac{i}{2f_2} \tau^{(13)} \zeta + \frac{D_a f_2}{2f_2} \tau^{(11)} \gamma^a \zeta
 \end{aligned} \tag{C.15}$$

C.2.2 The calculation of $\mathfrak{g} \mathcal{B}^{-1} \varepsilon^*$

The relevant expression is as follows,

$$\mathfrak{g} \mathcal{B}^{-1} \varepsilon^* = -\frac{3!}{96} e_B g_a (\Gamma^B \Gamma^{67a} + 2\Gamma^{67a} \Gamma^B) \mathcal{B}^{-1} \varepsilon^* \tag{C.16}$$

A few useful equations are as follows,

$$\begin{aligned}
 \Gamma^m \Gamma^{67b} + 2\Gamma^{67b} \Gamma^m &= -\Gamma^m \Gamma^{67b} \\
 \Gamma^i \Gamma^{67b} + 2\Gamma^{67b} \Gamma^i &= 3\Gamma^i \Gamma^{67b} \\
 \Gamma^a \Gamma^{67b} + 2\Gamma^{67b} \Gamma^a &= \Gamma^{67} (3\delta^{ab} - \Gamma^{ab}) = i(I_8 \otimes \gamma_{(2)} \otimes I_2) (3\delta^{ab} I_2 - \gamma^{ab})
 \end{aligned} \tag{C.17}$$

where $\gamma^{ab} = \varepsilon^{ab} \sigma^3$ and $\varepsilon^{89} = 1$. Projecting along the various directions we obtain,

$$\begin{aligned}
 (m) \quad & \Gamma_m \sum_{\eta_1, \eta_2} \chi^{\eta_1, \eta_2} \otimes \left(\frac{i}{16} g_a \gamma^a \star \zeta_{-\eta_1, \eta_2} \right) \\
 (i) \quad & \Gamma_i \sum_{\eta_1, \eta_2} \chi^{\eta_1, \eta_2} \otimes \left(-\frac{3i}{16} g_a \gamma^a \star \zeta_{-\eta_1, \eta_2} \right) \\
 (a) \quad & \sum_{\eta_1, \eta_2} \chi^{\eta_1, \eta_2} \otimes \left(-\frac{3i}{16} g_a \star \zeta_{\eta_1, -\eta_2} + \frac{i}{16} g_b \gamma^{ab} \star \zeta_{\eta_1, -\eta_2} \right)
 \end{aligned} \tag{C.18}$$

Using the τ -matrix notation, we can write the coefficient of $\Gamma_M \chi^{\eta_1, \eta_2}$ in the form,

$$\begin{aligned}
 (m) \quad & +\frac{i}{16} g_a \tau^{(10)} \gamma^a \star \zeta \\
 (i) \quad & -\frac{3i}{16} g_a \tau^{(10)} \gamma^a \star \zeta \\
 (a) \quad & -\frac{3i}{16} g_a \tau^{(01)} \star \zeta + \frac{i}{16} g_b \tau^{(01)} \gamma^{ab} \star \zeta
 \end{aligned} \tag{C.19}$$

C.2.3 Assembling the complete gravitino BPS equation

Now we combine the reduced gravitino equations. We again argue that the $\Gamma_M \chi^{\eta_1, \eta_2}$ are linearly independent which leads to the equations,

$$\begin{aligned}
 (m) \quad & 0 = \frac{1}{2f_6} \tau^{(30)} \zeta + \frac{D_a f_6}{2f_6} \tau^{(11)} \gamma^a \zeta + \frac{i}{16} g_a \tau^{(10)} \gamma^a \star \zeta \\
 (i) \quad & 0 = \frac{i}{2f_2} \tau^{(13)} \zeta + \frac{D_a f_2}{2f_2} \tau^{(11)} \gamma^a \zeta - \frac{3i}{16} g_a \tau^{(10)} \gamma^a \star \zeta \\
 (a) \quad & 0 = \left(D_a + \frac{i}{2} \hat{\omega}_a \sigma^3 - \frac{i}{2} q_a \right) \zeta - \frac{3i}{16} g_a \tau^{(01)} \star \zeta + \frac{i}{16} g_b \tau^{(01)} \gamma^{ab} \star \zeta
 \end{aligned} \tag{C.20}$$

where $\hat{\omega}_a = (\hat{\omega}_{89})_a$ is the spin connection along Σ . Eliminating the star using the definition (C.2), namely $\star \zeta = \tau^{(02)} \otimes \sigma^2 \zeta^*$, the system of gravitino BPS equations is then,

$$\begin{aligned}
 (m) \quad & 0 = \frac{1}{2f_6} \tau^{(30)} \zeta + \frac{D_a f_6}{2f_6} \tau^{(11)} \gamma^a \zeta + \frac{i}{16} g_a \tau^{(12)} \gamma^a \sigma^2 \zeta^* \\
 (i) \quad & 0 = \frac{i}{2f_2} \tau^{(13)} \zeta + \frac{D_a f_2}{2f_2} \tau^{(11)} \gamma^a \zeta - \frac{3i}{16} g_a \tau^{(12)} \gamma^a \sigma^2 \zeta^* \\
 (a) \quad & 0 = \left(D_a + \frac{i}{2} \hat{\omega}_a \sigma^3 - \frac{i}{2} q_a \right) \zeta + \frac{3}{16} g_a \tau^{(03)} \sigma^2 \zeta^* - \frac{1}{16} g_b \tau^{(03)} \gamma^{ab} \sigma^2 \zeta^*
 \end{aligned} \tag{C.21}$$

Upon multiplying the (m) and (i) equations by $\tau^{(11)}$, we find,

$$\begin{aligned}
 (m) \quad & 0 = -\frac{i}{2f_6} \tau^{(21)} \zeta + \frac{D_a f_6}{2f_6} \gamma^a \zeta - \frac{1}{16} g_a \tau^{(03)} \gamma^a \sigma^2 \zeta^* \\
 (i) \quad & 0 = \frac{1}{2f_2} \tau^{(02)} \zeta + \frac{D_a f_2}{2f_2} \gamma^a \zeta + \frac{3}{16} g_a \tau^{(03)} \gamma^a \sigma^2 \zeta^* \\
 (a) \quad & 0 = \left(D_a + \frac{i}{2} \hat{\omega}_a \sigma^3 - \frac{i}{2} q_a \right) \zeta + \frac{3}{16} g_a \tau^{(03)} \sigma^2 \zeta^* - \frac{1}{16} g_b \tau^{(03)} \gamma^{ab} \sigma^2 \zeta^*
 \end{aligned} \tag{C.22}$$

D Verifying the Bianchi identities

In this appendix we show that the reduced BPS equations summarized in section 4.3 imply that the Bianchi identities (2.1) are satisfied. In the complex coordinates introduced in (4.5), the Bianchi identities for P and Q take the following form,

$$\begin{aligned}\partial_w P_{\bar{w}} - \partial_{\bar{w}} P_w - 2iQ_w P_{\bar{w}} + 2iQ_{\bar{w}} P_w &= 0 \\ \partial_w Q_{\bar{w}} - \partial_{\bar{w}} Q_w + iP_w \bar{P}_{\bar{w}} - iP_{\bar{w}} \bar{P}_w &= 0\end{aligned}\quad (\text{D.1})$$

They are satisfied automatically if P and Q are expressed in terms of B as in (2.5). The Bianchi identity that needs to be checked is the one for G . We will find that (4.12) alone implies that it is satisfied. In terms of the lower-case expansion coefficients p, q, g defined in (2.16), the Bianchi identity (2.1) becomes,

$$\partial_w(\rho f_2^2 g_{\bar{z}}) - \partial_{\bar{w}}(\rho f_2^2 g_z) + \rho^2 f_2^2 (-iq_z g_{\bar{z}} + iq_{\bar{z}} g_z + p_z (g_z)^* - p_{\bar{z}} (g_{\bar{z}})^*) \quad (\text{D.2})$$

Using (4.1) and (2.5), this can be rewritten in terms of B, ρ and the κ_{\pm} alone as,

$$\begin{aligned}\left(\frac{a^2}{b^2} + e^{-i\vartheta}\right) \left[\Delta B + \frac{5}{2}f^2 \bar{B}(\partial_w B)(\partial_{\bar{w}} B)\right] + \frac{a^2}{b^2}(\partial_w B)\partial_{\bar{w}} \log \left[\frac{a(a\bar{a} - b\bar{b})^2}{\rho^2}\right] \\ + e^{-i\vartheta}(\partial_{\bar{w}} B)\partial_w \log \left[\frac{\bar{b}(a\bar{a} - b\bar{b})^2}{\rho^2}\right] + \left(\frac{3}{2}B + e^{i\vartheta}\right) f^2 \left(\frac{a^2}{b^2}|\partial_w B|^2 + e^{-i\vartheta}|\partial_{\bar{w}} B|^2\right) = 0\end{aligned}\quad (\text{D.3})$$

where $\Delta = \partial_w \partial_{\bar{w}}$ and,

$$\begin{aligned}\bar{a}^2 &= \rho \bar{\alpha}^2 / f = \kappa_+ + B\kappa_- \\ \bar{b}^2 &= \rho \bar{\beta}^2 / f = \bar{B}\kappa_+ + \kappa_-\end{aligned}\quad (\text{D.4})$$

are used as shorthands for the corresponding combinations of κ_{\pm} . The only place where ρ^2 shows up is in the log derivatives, where it can be eliminated using (4.12), leaving an expression solely in terms of B and \bar{B} .

To show that (4.12) implies that the Bianchi identity is satisfied, we first eliminate the two-derivative term. To this end, we act on (4.14) with ∂_w and on the complex conjugate of (4.14) with $\partial_{\bar{w}}$. The result are two equations linear in ΔB and $\Delta \bar{B}$, namely,

$$\begin{aligned}\bar{X} \Delta B - \bar{Y} \Delta \bar{B} + (\partial_w B)\partial_{\bar{w}} \bar{X} - (\partial_w \bar{B})\partial_{\bar{w}} \bar{Y} &= (\partial_w f^{-4})\kappa_+^2 \partial_w \frac{\kappa_-}{\kappa_+} \\ X \Delta \bar{B} - Y \Delta B + (\partial_{\bar{w}} \bar{B})\partial_w X - (\partial_{\bar{w}} B)\partial_w Y &= (\partial_w f^{-4})\bar{\kappa}_+^2 \partial_w \frac{\bar{\kappa}_-}{\bar{\kappa}_+}\end{aligned}\quad (\text{D.5})$$

where $X = b^4 + 2a^2 b^2 e^{i\theta}$ and $Y = a^4 + 2a^2 b^2 e^{-i\theta}$. Eliminating $\Delta \bar{B}$ yields an equation for ΔB in terms of first derivatives, namely,

$$\begin{aligned}\Delta B(|X|^2 - |Y|^2) &= X(\partial_w \bar{B})\partial_{\bar{w}} \bar{Y} + \bar{Y}(\partial_{\bar{w}} B)\partial_w Y - X(\partial_w B)\partial_{\bar{w}} \bar{X} - \bar{Y}(\partial_{\bar{w}} \bar{B})\partial_w X \\ &\quad + X(\partial_{\bar{w}} f^{-4})\kappa_+^2 \partial_w \frac{\kappa_-}{\kappa_+} + \bar{Y}(\partial_w f^{-4})\bar{\kappa}_+^2 \partial_w \frac{\bar{\kappa}_-}{\bar{\kappa}_+}\end{aligned}\quad (\text{D.6})$$

Substituting this expression in (D.3) leaves only first derivatives. We can now use (4.14) and its complex conjugate to eliminate the derivatives of \bar{B} . After this step the left hand side of (D.3) collapses to zero, showing that the Bianchi identity is satisfied.

E Supergravity fields in terms of holomorphic data

In this appendix we derive the expression for the supergravity fields from the local solution to the BPS equations derived in section 4. The results are summarized in section 5. As a prerequisite, we note that (5.7) arises from (4.59) with,

$$\kappa_- \bar{\mathcal{L}} = -\partial_w \mathcal{G} \qquad \kappa^2 = -\partial_w \partial_{\bar{w}} \mathcal{G} \quad (\text{E.1})$$

We then start with the metric functions and the axion-dilaton combination, before turning to the flux field.

E.1 Metric functions

We start with the expressions for the metric functions f_2 , f_6 in terms of α , β given in (3.46). Using the expressions for α , β in terms of κ_{\pm} and f given in (4.9) yields

$$\left(f_6 + \frac{3}{\nu} f_2\right)^2 = \frac{4c_6^2 f^2}{\rho^2} |\kappa_-|^2 |B\bar{\lambda} + 1|^2 \qquad \left(f_6 - \frac{3}{\nu} f_2\right)^2 = \frac{4c_6^2 f^2}{\rho^2} |\kappa_-|^2 |\lambda + B|^2 \quad (\text{E.2})$$

Translating to Z and using (4.22) for f^2 yields

$$f_6 + \frac{3}{\nu} f_2 = \frac{2c_6}{\rho} \sqrt{\frac{\kappa^2}{1 - |Z|^4}} \qquad f_6 - \frac{3}{\nu} f_2 = \frac{2c_6}{\rho} |Z|^2 \sqrt{\frac{\kappa^2}{1 - |Z|^4}} \quad (\text{E.3})$$

Solving for f_2 and f_6 yields

$$f_2 = \frac{\nu c_6}{3\rho} \sqrt{\kappa^2 \frac{1 - |Z|^2}{1 + |Z|^2}} \qquad f_6 = \frac{c_6}{\rho} \sqrt{\kappa^2 \frac{1 + |Z|^2}{1 - |Z|^2}} \quad (\text{E.4})$$

With $|Z|^2 = R$ and $\nu^2 = 1$, this immediately leads to the expressions in (5.8). Next comes ρ^2 , the metric on Σ . From (4.27), (4.36) and (4.48),

$$\hat{\rho}^2 = \frac{\rho^2}{|\kappa_-|^2} \frac{(1 - |Z|^2)^{3/2}}{|Z| \sqrt{1 - |\lambda|^2} \sqrt{1 + |Z|^2}} \qquad \hat{\rho}^4 = \frac{1}{\xi \bar{\xi}} = \frac{(1 - \lambda \bar{\lambda})^2}{\mathcal{L} \bar{\mathcal{L}}} \quad (\text{E.5})$$

From those one finds

$$\rho^2 = \frac{\kappa^2}{|\kappa_- \bar{\mathcal{L}}|} \frac{|Z|}{1 - |Z|^2} \sqrt{\kappa^2 \frac{1 + |Z|^2}{1 - |Z|^2}} \quad (\text{E.6})$$

With $|Z|^2 = R$ and $\kappa_- \bar{\mathcal{L}} = -\partial_w \mathcal{G}$, this directly leads to the expression quoted in (5.8).

To get to the axion and dilaton we work out the expression for B . Using the definition in (4.21) and (4.27)

$$B = \frac{R e^{i\psi} - \lambda}{1 - \bar{\lambda} R e^{i\psi}} = \frac{R \bar{\mathcal{L}} - \lambda \mathcal{L}}{\mathcal{L} - \bar{\lambda} R \bar{\mathcal{L}}} \quad (\text{E.7})$$

The second equality uses $e^{i\psi} = \bar{\xi}/\xi$ and (4.48). Multiplying numerator and denominator by $|\kappa_-|^2$ yields

$$B = \frac{\bar{\kappa}_- (R \kappa_- \bar{\mathcal{L}}) - \kappa_+ (\bar{\kappa}_- \mathcal{L})}{\kappa_- (\bar{\kappa}_- \mathcal{L}) - \bar{\kappa}_+ (R \kappa_- \bar{\mathcal{L}})} \quad (\text{E.8})$$

The explicit expression for f^2 can be obtained from (4.22) as,

$$f^2 = \frac{|\kappa_-|^2}{\kappa_-^2 |\mathcal{L}|^2} \frac{|\bar{\mathcal{L}} - \lambda R \mathcal{L}|^2}{1 - R^2}. \quad (\text{E.9})$$

For τ we find

$$i\tau = \frac{\lambda_+ - \bar{\lambda}_+ Z^2}{\lambda_- + \bar{\lambda}_- Z^2} = \frac{\lambda_+ \mathcal{L} - \bar{\lambda}_+ \bar{\mathcal{L}} R}{\lambda_- \mathcal{L} + \bar{\lambda}_- \bar{\mathcal{L}} R} \quad \lambda_{\pm} = \lambda \pm 1 \quad (\text{E.10})$$

To get to the axion and dilaton we separate the real and imaginary parts and use $\tau = \chi + ie^{-2\phi}$, which yields

$$e^{-2\phi} = \frac{\kappa_-^2 |\mathcal{L}|^2}{D |\kappa_-|^2} \frac{1 - R^2}{R} \quad D = |\lambda_- \mathcal{L}|^2 W + \lambda_-^2 \mathcal{L}^2 + \bar{\lambda}_-^2 \bar{\mathcal{L}}^2 \quad (\text{E.11})$$

The axion becomes

$$\chi = i \frac{(\lambda - \bar{\lambda}) |\mathcal{L}|^2 W - \lambda_+ \lambda_- \mathcal{L}^2 + \bar{\lambda}_+ \bar{\lambda}_- \bar{\mathcal{L}}^2}{D} \quad (\text{E.12})$$

E.2 Flux field

The complex 3-form field $G_{(3)}$ is better expressed in terms of the complex 3-form field $F_{(3)}$ by the first relation of (2.6). Since $F_{(3)}$ is a closed 3-form it may locally be written in terms of a complex flux potential 2-form field $C_{(2)}$ by $F_{(3)} = dC_{(2)}$. Inverting this relation, we have,

$$dC_{(2)} = f(G_{(3)} + B\bar{G}_{(3)}) \quad (\text{E.13})$$

The symmetries of the problem force $C_{(2)}$ and $G_{(3)}$ to be of the following form,

$$\begin{aligned} C_{(2)} &= \mathcal{C} \hat{e}^{67} \\ G_{(3)} &= g_a e^a \wedge e^{67} = f_2^2 g_a e^a \wedge \hat{e}^{67} \end{aligned} \quad (\text{E.14})$$

so that,

$$\begin{aligned} \partial_w \mathcal{C} &= \rho f_2^2 f(g_z + B\bar{g}_z) \\ \partial_{\bar{w}} \mathcal{C} &= \rho f_2^2 f(g_{\bar{z}} + B\bar{g}_{\bar{z}}) \end{aligned} \quad (\text{E.15})$$

Using the conversion of G into P and then into derivatives of B using (2.5) and (4.1),

$$\begin{aligned} \rho g_z &= 4i \frac{\alpha}{\beta} f^2 \partial_w B & \rho g_{\bar{z}} &= -4i \frac{\bar{\beta}}{\alpha} f^2 \partial_{\bar{w}} B \\ \rho \bar{g}_z &= 4i \frac{\beta}{\alpha} f^2 \partial_w \bar{B} & \rho \bar{g}_{\bar{z}} &= -4i \frac{\bar{\alpha}}{\beta} f^2 \partial_{\bar{w}} \bar{B} \end{aligned} \quad (\text{E.16})$$

we obtain the following expressions,

$$\begin{aligned} \partial_w \mathcal{C} &= 4i f_2^2 f^3 \left(\frac{\alpha}{\beta} \partial_w B + B \frac{\beta}{\alpha} \partial_w \bar{B} \right) \\ \partial_w \bar{\mathcal{C}} &= 4i f_2^2 f^3 \left(\frac{\beta}{\alpha} \partial_w \bar{B} + \bar{B} \frac{\alpha}{\beta} \partial_w B \right) \end{aligned} \quad (\text{E.17})$$

We will now work towards expressing the right side in terms of the solutions to the BPS equations, and then integrating the equations to obtain \mathcal{C} and its complex conjugate.

E.2.1 Expressing variables in terms of holomorphic functions

Recall that we have,

$$\frac{\alpha}{\beta} = \left(\frac{\bar{\lambda} + \bar{B}}{\bar{\lambda}B + 1} \right)^{\frac{1}{2}} = \bar{Z} \left(\frac{1 - \bar{\lambda}Z^2}{1 - \lambda\bar{Z}^2} \right)^{\frac{1}{2}} \quad \left| \frac{\alpha}{\beta} \right| = |Z| \quad (\text{E.18})$$

This allows us to compute the combinations,

$$\begin{aligned} \frac{\alpha^2}{\beta^2} \partial_w B + B \partial_{\bar{w}} \bar{B} &= \frac{\bar{Z}^2(1 - |\lambda|^2)}{|1 - \lambda\bar{Z}^2|^2} \partial_w Z^2 + \frac{(Z^2 - \lambda)(1 - |\lambda|^2)}{(1 - \lambda\bar{Z}^2)|1 - \lambda\bar{Z}^2|^2} \partial_w \bar{Z}^2 \\ &\quad - \frac{\bar{Z}^2(1 - |\lambda|^2)(1 - |Z|^4)}{(1 - \lambda\bar{Z}^2)|1 - \lambda\bar{Z}^2|^2} \partial_w \lambda \\ \frac{\beta^2}{\alpha^2} \partial_w \bar{B} + \bar{B} \partial_w B &= \frac{1 - |\lambda|^2}{|1 - \lambda\bar{Z}^2|^2} \left(\frac{\partial_w \bar{Z}^2}{\bar{Z}^2} + \frac{\bar{Z}^2 - \bar{\lambda}}{1 - \bar{\lambda}Z^2} \partial_w Z^2 \right) \end{aligned} \quad (\text{E.19})$$

In the second equation, the terms proportional to $\partial_w \lambda$ cancel. Putting all together, we have,

$$\begin{aligned} \partial_w \mathcal{C} &= \frac{4ic_6^2}{9\hat{\rho}^2} \frac{(1 - |Z|^2)}{(1 + |Z|^2)^3} \frac{1}{\bar{Z}|Z|} \left(\partial_w |Z|^4 - \lambda(\partial_w \bar{Z}^2 + \bar{Z}^4 \partial_w Z^2) - \bar{Z}^2(1 - |Z|^4) \partial_w \lambda \right) \\ \partial_w \bar{\mathcal{C}} &= \frac{4ic_6^2}{9\hat{\rho}^2} \frac{(1 - |Z|^2)}{(1 + |Z|^2)^3} \frac{1}{\bar{Z}|Z|} \left(-\bar{\lambda} \partial_w |Z|^4 + \partial_w \bar{Z}^2 + \bar{Z}^4 \partial_w Z^2 \right) \end{aligned} \quad (\text{E.20})$$

Next, we eliminate $\hat{\rho}^2$ in favor of ξ , by using the following rearrangement formula,

$$\frac{1}{\hat{\rho}^2 \bar{Z}|Z|} = \frac{\bar{\xi}}{|Z|^2} \quad (\text{E.21})$$

so that we find,

$$\begin{aligned} \partial_w \mathcal{C} &= \frac{4ic_6^2}{9} \frac{(1 - |Z|^2)}{(1 + |Z|^2)^3} \frac{\bar{\xi}}{|Z|^2} \left(\partial_w |Z|^4 - \lambda(\partial_w \bar{Z}^2 + \bar{Z}^4 \partial_w Z^2) - \bar{Z}^2(1 - |Z|^4) \partial_w \lambda \right) \\ \partial_w \bar{\mathcal{C}} &= \frac{4ic_6^2}{9} \frac{(1 - |Z|^2)}{(1 + |Z|^2)^3} \frac{\bar{\xi}}{|Z|^2} \left(-\bar{\lambda} \partial_w |Z|^4 + \partial_w \bar{Z}^2 + \bar{Z}^4 \partial_w Z^2 \right) \end{aligned} \quad (\text{E.22})$$

Instead of working with \mathcal{C} and $\bar{\mathcal{C}}$, we form the combination $\bar{\mathcal{C}} + \bar{\lambda}\mathcal{C}$,

$$\partial_w (\bar{\mathcal{C}} + \bar{\lambda}\mathcal{C}) = \frac{4ic_6^2}{9} \mathcal{P} \quad (\text{E.23})$$

where \mathcal{P} is given by,

$$\mathcal{P} = \xi \frac{1 - |Z|^2}{(1 + |Z|^2)^3} \left((1 - |\lambda|^2)(\partial_w \ln \bar{Z}^2 + \bar{Z}^2 \partial_w Z^2) + (1 - |Z|^4) \partial_w (1 - |\lambda|^2) \right) \quad (\text{E.24})$$

Having $\bar{\mathcal{C}} + \bar{\lambda}\mathcal{C}$ is essentially as good as having \mathcal{C} itself, since from $\bar{\mathcal{C}} + \bar{\lambda}\mathcal{C}$ and its complex conjugate we may obtain \mathcal{C} and $\bar{\mathcal{C}}$. Next, we use the formula $\xi(1 - |\lambda|^2) = \mathcal{L}$ to eliminate ξ in favor of λ and \mathcal{L} ,

$$\mathcal{P} = \mathcal{L} \frac{1 - |Z|^2}{(1 + |Z|^2)^3} \left(\partial_w \ln \bar{Z}^2 + \bar{Z}^2 \partial_w Z^2 + (1 - |Z|^4) \partial_w \ln(1 - |\lambda|^2) \right) \quad (\text{E.25})$$

Changing variables from Z to $Z^2 = Re^{i\psi}$, we find,

$$\mathcal{P} = \mathcal{L} \frac{(1-R)(R^2+1)}{(1+R)^3} \frac{\partial_w R}{R} - \mathcal{L} \frac{(R-1)^2}{(R+1)^2} \left(i\partial_w \psi - \partial_w \ln(1-|\lambda|^2) \right) \quad (\text{E.26})$$

Expressing this combination in terms of $W = R + R^{-1}$, we find,

$$\mathcal{P} = -\mathcal{L} \frac{W}{(W+2)^2} \partial_w W - \mathcal{L} \frac{W-2}{W+2} \left(i\partial_w \psi - \partial_w \ln(1-|\lambda|^2) \right) \quad (\text{E.27})$$

Using equation (4.52), we eliminate $\hat{\rho}$ in favor of ξ , and eliminate ξ in favor of \mathcal{L} and λ ,

$$\partial_w W = -(W+1)\partial_w \ln \mathcal{L} \bar{\mathcal{L}} + (W-2)\partial_w \ln(1-|\lambda|^2) + 3i\partial_w \psi \quad (\text{E.28})$$

Putting all together, we have,

$$\begin{aligned} \mathcal{P} = \frac{\mathcal{L}}{(W+2)^2} & \left(W(W+1)\partial_w \ln \mathcal{L} \bar{\mathcal{L}} + 2(W-2)\partial_w \ln(1-|\lambda|^2) \right. \\ & \left. -(W^2+3W-4)\partial_w \ln \frac{\bar{\mathcal{L}}}{\mathcal{L}} \right) \end{aligned} \quad (\text{E.29})$$

where we have used the relation $e^{i\psi} = \bar{\mathcal{L}}/\mathcal{L}$ to express the derivative $i\partial_w \psi$ in terms of \mathcal{L} . Finally, W is given in terms of holomorphic data by,

$$W = 2 + 6 \frac{1-|\lambda|^2}{\mathcal{L}\bar{\mathcal{L}}} \left(\mathcal{A}_+ \bar{\mathcal{A}}_+ - \mathcal{A}_- \bar{\mathcal{A}}_- + \mathcal{B} + \bar{\mathcal{B}} \right) \quad (\text{E.30})$$

Separating the dependence of \mathcal{L} and $\bar{\mathcal{L}}$, we find,

$$\mathcal{P} = \frac{2\mathcal{L}}{(W+2)^2} \left((W^2+2W-2)\partial_w \ln \mathcal{L} + (W-2) \left\{ \partial_w \ln(1-|\lambda|^2) - \partial_w \ln \bar{\mathcal{L}} \right\} \right) \quad (\text{E.31})$$

Next, we return to the differential equation satisfied by W in (E.28) and notice that the precise same combination $(W-2)\partial_w \ln(1-|\lambda|^2)$ occurs there. Expressing ψ in terms of \mathcal{L} in this equation, it takes the form,

$$(W-2) \left\{ \partial_w \ln(1-|\lambda|^2) - \partial_w \ln \bar{\mathcal{L}} \right\} = \partial_w W + (W+4)\partial_w \ln \mathcal{L} \quad (\text{E.32})$$

Thus, we will proceed to eliminate this term between (E.28) and (E.31), and we find,

$$\mathcal{P} = \frac{2(W+1)}{W+2} \partial_w \mathcal{L} + \frac{2\mathcal{L}}{(W+2)^2} \partial_w W \quad (\text{E.33})$$

It is straightforward to integrate this equation, and we have,

$$\mathcal{P} = \partial_w \left(\frac{2\mathcal{L}(W+1)}{W+2} \right) \quad (\text{E.34})$$

Therefore, we have,

$$\bar{\mathcal{C}} + \bar{\lambda}\mathcal{C} = \bar{\mathcal{K}}_1 + \frac{8ic_6^2}{9} \frac{\mathcal{L}(W+1)}{W+2} \quad (\text{E.35})$$

for an as yet undetermined holomorphic function \mathcal{K}_1 .

Proceeding analogously for $\bar{\mathcal{C}}$, and using the equations for $\partial_w \xi$ and $\partial_w \bar{\xi}$, we find,

$$\partial_w \bar{\mathcal{C}} = \frac{4ic_6^2}{9} \partial_w \left(-2 \frac{\xi + \bar{\lambda} \bar{\xi}}{W+2} + \frac{\mathcal{L}}{1-|\lambda|^2} + \mathcal{A}_- \right) \quad (\text{E.36})$$

so that

$$\bar{\mathcal{C}} = \frac{4ic_6^2}{9} \left(-2 \frac{\xi + \bar{\lambda} \bar{\xi}}{W+2} + \frac{\mathcal{L}}{1-|\lambda|^2} + \mathcal{A}_- + \bar{\mathcal{K}}_2 \right) \quad (\text{E.37})$$

for some holomorphic function \mathcal{K}_2 . Equating now the two different expressions for $\bar{\mathcal{C}} + \bar{\lambda} \mathcal{C}$, we get after some simplifications which eliminate the dependence on W completely,

$$\bar{\mathcal{K}}_1 - \bar{\mathcal{A}}_+ - \bar{\mathcal{K}}_2 + \bar{\lambda}(3\bar{\mathcal{A}}_- - 2\mathcal{A}_+ + \mathcal{K}_2) = 0 \quad (\text{E.38})$$

Separating holomorphic and anti-holomorphic dependences, we find,

$$\mathcal{K}_1 = 3\mathcal{A}_+ - 3\lambda\mathcal{A}_- + \mathcal{K}_0 + \bar{\mathcal{K}}_0\lambda \quad (\text{E.39})$$

where \mathcal{K}_0 is an arbitrary complex constant. Thus, we have,

$$\mathcal{C} = \frac{4ic_6^2}{9} \left(+2 \frac{\bar{\xi} + \lambda \xi}{W+2} - \frac{\bar{\mathcal{L}}}{1-|\lambda|^2} - \bar{\mathcal{A}}_- - 2\mathcal{A}_+ - \mathcal{K}_0 \right) \quad (\text{E.40})$$

Evaluating that further gives

$$\mathcal{C} = \frac{4ic_6^2}{9} \left(\frac{2\lambda\mathcal{L} - W\bar{\mathcal{L}}}{(W+2)(1-|\lambda|^2)} - \bar{\mathcal{A}}_- - 2\mathcal{A}_+ - \mathcal{K}_0 \right) \quad (\text{E.41})$$

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